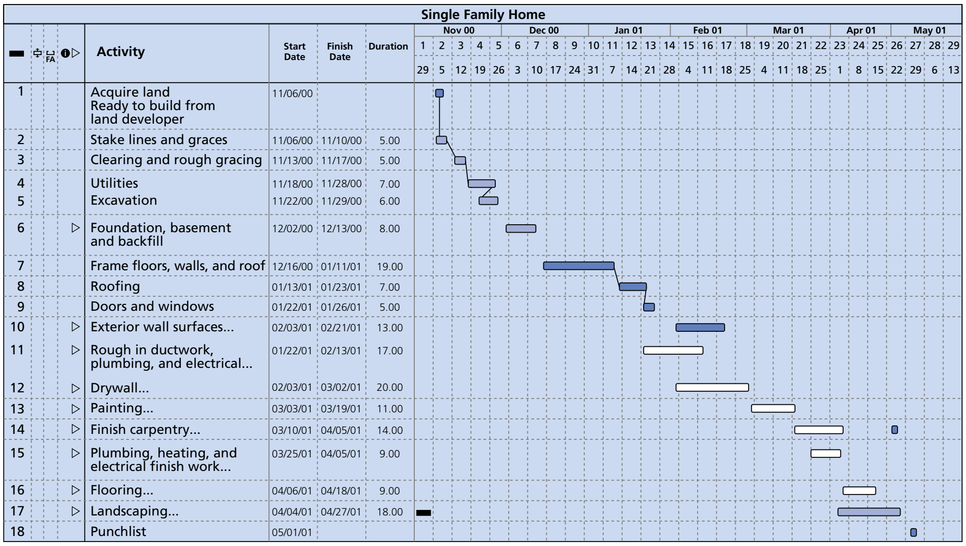
# 6.5 Solving Problems with Graphs

When planning the construction of a house, a contractor has to schedule a complicated series of events. Land has to be purchased, and plans have to be finalized and approved by the municipal building department. Carpenters, plumbers, electricians, and other skilled trades have to be hired. Materials have to be delivered and equipment has to be available on site when they are needed.

Many builders use a graphical tool called a Gantt chart to help them schedule the activities, equipment, materials, and personnel in the most efficient and least costly manner. A sample Gantt chart is shown below.




This Gantt chart shows a simplified version of the activities that must occur, the order in which they must take place, and the duration of each activity. Some of the tasks listed depend on other activities being completed before they can begin. These tasks are called **sequential tasks**. For example, the land must be purchased before anything else can occur. The foundation cannot be poured until the hole for the foundation has been excavated.

**sequential tasks**—tasks that are dependent on each other and must be performed in order

**project graph**—a digraph on which each task that must be completed as part of the project is a vertex. Vertices that are dependent are connected with an edge

**weight of an edge**—the duration a preceding task takes to complete

**parallel tasks**—tasks that are independent and can be performed at the same time

Tasks that are dependent in this way have a directed line segment that links them on a **project graph**. Often there is a number, or **weight**, associated with each edge. This typically represents the duration of the preceding task.

On the other hand, some activities are independent of one another. Landscaping, for example, can occur at the same time that the drywall is being installed. These tasks are called **parallel tasks**.

The tasks required to complete the construction of a home can be extensive. The chart above was created with a specialized computer program called *FastTrack Schedule*. This software allows contractors to manage the resources and people, and also allows them to track the cost of the project so they can stay within their budget.

## CRITICAL PATH ANALYSIS

In the middle of the twentieth century, the U.S. government developed methods to monitor complex military construction projects. The branch of mathematics that evolved became known as Operations Research. One of the tasks of Operations Research was **critical path analysis**, which was involved in identifying tasks in a process that were critical to the earliest completion of the project.

One of the most useful applications of critical path analysis is determining the **earliest start time (EST)** for each task. By comparing the weights of each edge in a project graph, you can find the time at which all prerequisite tasks have been completed (the earliest start time).

**critical path analysis**—finding the optimal way to complete a complex task

**earliest start time**—the earliest that a task can begin if all the tasks on which it depends begin as early as possible

**task table**—a table that lists all the tasks associated with a project, including the duration of each task and the tasks on which each depends

### Example 1 Using Graphs to Schedule a Project

Using the **task table** shown below, determine the minimum amount of time required to complete the construction of a home. (The prerequisite tasks are those jobs that need to be completed before the indicated task can start.)

**Project Connection**
Setting up a task table for your final project will help you determine all the tasks you need to complete.

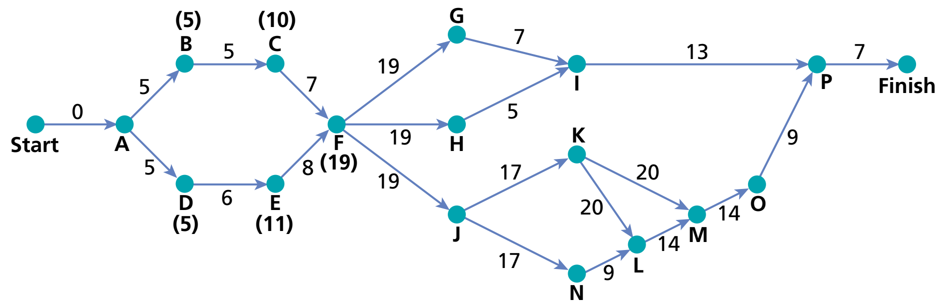
| Task | Description | Duration (days) | Prerequisite Task(s) |
|---|---|---|---|
| Start | Acquire land | 0 | |
| A | Stake lines and grades | 5 | none |
| B | Clear site and do rough grading | 5 | A |
| C | Install service for all utilities | 7 | B |
| D | Excavate | 6 | A |
| E | Lay foundation, pour basement, and then backfill | 8 | D |
| F | Frame doors, walls, and roof | 19 | C, E |
| G | Shingle roof | 7 | F |
| H | Install doors and windows | 5 | F |
| I | Build exterior wall surfaces | 13 | G, H |
| J | Rough-in ductwork, plumbing, and electrical | 17 | F |
| K | Install drywall | 20 | J |
| L | Paint | 14 | K, N |
| M | Do finish carpentry | 14 | K, L |
| N | Do plumbing, heating, and electrical work | 9 | J |
| O | Install flooring | 9 | M |
| P | Complete legal work to transfer ownership | 7 | I, O |
| Finish | | | |

## Solution

Drawing a project graph of the various tasks makes it easier to see how the tasks relate to one another. Each vertex corresponds to one of the tasks. The arrow on each edge indicates that a task depends on another to be completed before it can start. The number on the edge is the time required for the preceding task to be completed.

**Think about**
**Graphic Solutions**

Vertex F corresponds to framing the doors, walls, and roof. From the graph, what is the earliest that Task F can be completed after the project is begun?




In calculating the EST, we can see that

- Task B depends on Task A being completed, so (5) is indicated above B for 5 days on the graph
- Task C requires Task B to be completed, so (10) is indicated above C (5 + 5 = 10)
- Task E requires Task D to be completed, so (11) is indicated below E (5 + 6 = 11)
- Task F cannot begin until 19 days have elapsed because both Tasks C and E must be completed first (11 + 8 = 19)

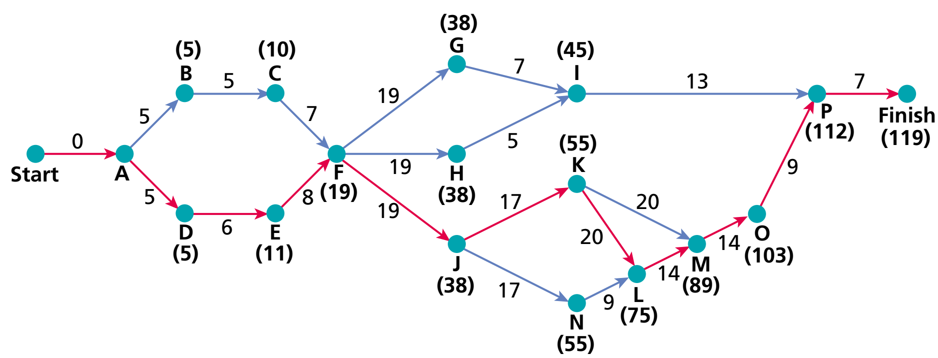
**Think about**
**Earliest Start Time**

The first few ESTs are shown on the graph. Complete the graph before reading the rest of this page.

The least number of days in which the house can be completed is the time it takes to complete all the tasks in the project. This corresponds to the total time represented by the longest path from start to finish on the project graph. That path is shown in red on the following graph. The earliest the job can be completed is 119 days.

The critical path for this project is

Start – A – D – E – F – J – K – L – M – O – P – Finish.



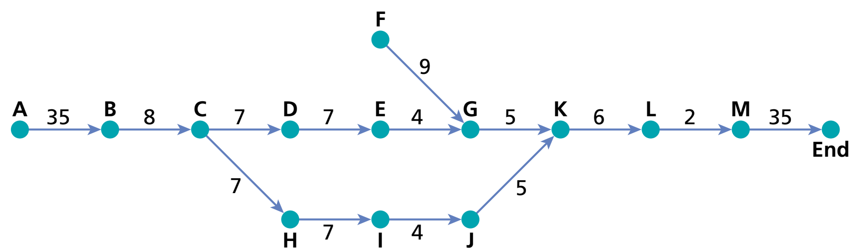
## Example 2 Using an ATM

Each week Vasundhara deposits her paycheque, keeping $40 for herself. In an effort to be more organized, she times each part of her weekly routine at the bank machine and creates the following task table.

| Task | Description | Time (s) | Prerequisite |
|---|---|---|---|
| A | Enter bank | 35 | |
| B | Insert bank card | 8 | A |
| C | Type in PIN | 7 | B |
| D | Select deposit | 7 | C |
| E | Key in amount | 4 | D |
| F | Put cheque in envelope | 9 | |
| G | Insert envelope | 5 | E, F |
| H | Select withdrawal | 7 | C |
| I | Type in amount | 4 | H |
| J | Collect money | 5 | I |
| K | Press DONE | 6 | G, J |
| L | Retrieve bank card | 2 | K |
| M | Exit bank | 35 | L |

Create a project graph and use critical path analysis to determine the optimal order in which to perform this transaction. Find the EST.

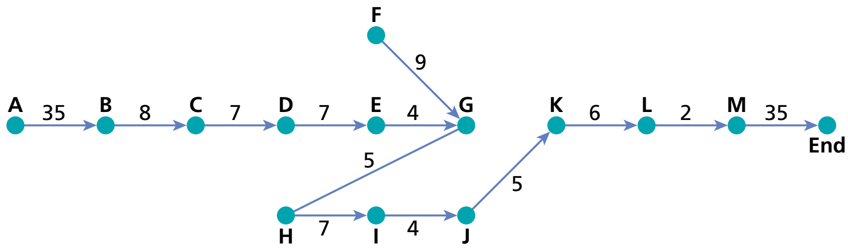
### Solution




After creating a graph, Vasundhara notices the following facts:

- As long as she can remember the amount the cheque is made out for, Task F can be done any time.
- Since her account balance is greater than $40 on a regular basis, she can do either the deposit or the withdrawal first.
- The graph makes a critical error. It implies that Tasks D, E, and G can be done at the same time as H, I, and J. While the order doesn't matter, Vasundhara must choose one of the two series of tasks to do first.

Vasundhara decides to streamline her transaction. She decides to have her cheque already in an envelope and will deposit it first. Her corrected graph is shown below.




The minimum amount of time the transaction will take is 125 s (2 min, 5 s), assuming there is no line-up!

### KEY IDEAS

**task table**—a table that lists all the tasks required to complete a project. For each task entry, it includes the duration of each task and a list of tasks that must be finished before that task may be started.

**sequential tasks**—tasks that are dependent on each other and must be performed in order

**project graph**—a digraph on which each task that must be completed as part of the project is a vertex. Task vertices are joined to those vertices that depend on their prior completion.

**weight of an edge**—the number associated with an edge; typically represents the time needed to accomplish the preceding task

**parallel tasks**—tasks that are independent and can be performed at the same time

**critical path analysis**—finding the optimal way to complete a complex task

**earliest start time**—the time required for the latest completion of its prerequisite tasks

**critical path**—the longest path (in terms of completion times) from start to finish through the project graph

**graph theory**—the study of graphs and their application in solving puzzles

## 6.5 Exercises

**A**

**1.** Take the following sequential tasks and put them in an appropriate order.
- **(a)** click OK, turn off computer, turn on computer, click on Shortcut, turn on printer, enter password, exit program, click Start, turn off printer, click Shut Down, click Print, turn on monitor, turn off monitor
- **(b)** get plate from cupboard, get two slices of bread, push down toaster switch, spread peanut butter on bread, get knife from drawer, get peanut butter from shelf, eat peanut butter on toast
- **(c)** go to third class, drive to school, go to fourth class, eat lunch, park the car, drive home, go to second class, go to first class, go to after-school practice
- **(d)** select CD, press Play, turn on stereo receiver, adjust volume, turn on CD player

**2.** For each list in Question 1, create a task table listing the dependencies.

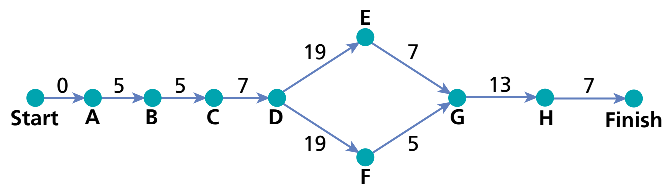
**3.** **Communication** Explain the difference between parallel tasks and sequential tasks. Give two examples of each.

**B**

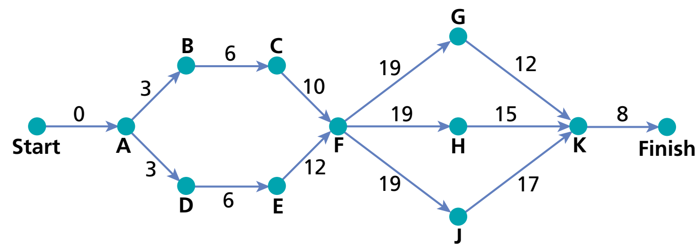
**4.** List, in order, five sequential tasks that are part of washing a car.

**5.** **Knowledge and Understanding** For each of the project graphs below, do the following. (Assume all times are in days.)
- **(i)** Complete the graph by adding the earliest start times for each task.
- **(ii)** Identify the critical path for the project.
- **(iii)** Determine the least amount of time required for the project.

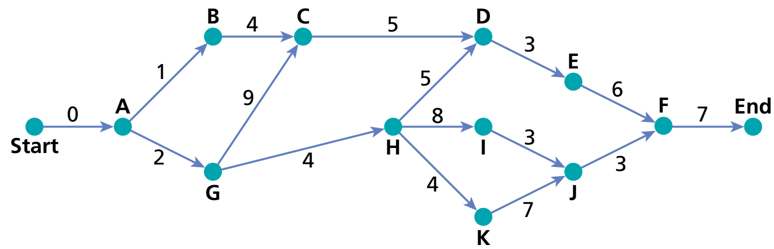
**(a)**



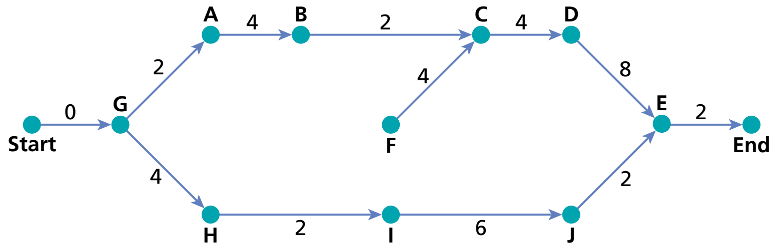

**(b)**



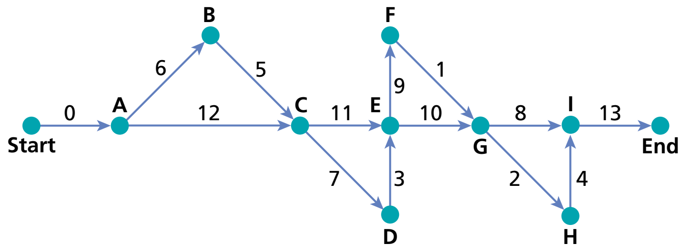

**(c)**



(d)




(e)




**6.** **Application** A shipyard is working on the construction of a passenger ship.

| Task | Task Description | Duration (in months) | Prerequisite Task(s) |
|---|---|---|---|
| A | Create initial design | 1 | none |
| B | Prepare detailed design | 2.5 | A |
| C | Order steel | 1.5 | A |
| D | Lay keel | 1 | B, C |
| E | Build hull | 2 | D |
| F | Order and build engines | 2 | B, C |
| G | Build superstructure | 2 | B, C |
| H | Install engines | 2 | E, F |
| I | Order and build navigation equipment | 2.5 | B |
| J | Install superstructure | 1 | G |
| K | Install navigation and other equipment | 1 | I, J |
| L | Trial runs | 1.5 | H, K |
| M | Install furniture, heating, lighting, etc. | 2.5 | J |

**(a)** Construct a project graph for this project.
**(b)** Determine the least time in which the project can be completed.

**7.** A bicycle is to be completely dismantled and carefully packaged prior to shipping it to an overseas buyer. The following table provides estimates of the time required to remove or disassemble each part. Determine the least completion time for this project.

| Task | Description | Time (min) | Prerequisites |
|---|---|---|---|
| A | chain | 5 | I |
| B | crank | 30 | A, I |
| C | fork | 35 | G, E |
| D | frame | 30 | C, F, J, K, N |
| E | front brake | 10 | G |
| F | front gear sprocket | 10 | A, B, G |
| G | front wheel | 5 | I |
| H | hand control levers | 10 | E |
| I | pedals | 20 | none |
| J | rear brake | 10 | L |
| K | rear gear sprocket | 20 | A, B, L |
| L | rear wheel | 5 | A |
| M | saddle | 1 | none |
| N | handle bars | 5 | H |

C

**8.** **Thinking, Inquiry, Problem Solving** Create a task table and project graph for all the tasks you need to complete prior to the final presentation of your course project. Use critical path analysis to determine the earliest completion time for your project.

**9.** Consider the following recipe.

**Mexican Party Pie**

Put 2 lb of ground beef in a frying pan and fry it until it is browned (about 10 min). Then mix in 2 tbsp of vegetable oil, $\frac{1}{2}$ of a chopped onion, 1 clove of garlic, 14 oz. of tomato sauce, 19 oz of kernel corn, 1 chopped green pepper, 1 tsp of sugar, 1 tsp of salt, and 1 tsp of chili powder. Stir for 5 min and pour into a $13 \times 9$ inch baking dish. Add topping and put it into the oven for 45 min.

To make the topping, combine 1 cup of cornmeal with $\frac{3}{4}$ cup of milk, mix, and let stand for 10 min. While you are waiting, mix $1\frac{1}{4}$ cups of flour with 1 tsp of salt, $2\frac{1}{2}$ tsp baking powder, and $\frac{1}{2}$ cup of sugar. Stir for 2 min and then add $\frac{1}{2}$ cup of vegetable shortening, 1 egg, and $\frac{3}{4}$ cup of milk. Mix for 5 min, and then add the cornmeal.

Make sure to preheat the oven to 375°F.

Assuming it takes 2 min to chop one kind of vegetable, 9 min for the oven to heat up, and 15 min for the dish to cool after baking,

**(a)** create a task table that lists the prerequisite tasks
**(b)** use your task table to create a project graph with weighted edges
**(c)** use the critical path to determine how long it will take to prepare this dish from start to serve

## ADDITIONAL ACHIEVEMENT CHART QUESTIONS

**10.** **Knowledge and Understanding** Consider the following list of tasks associated with a male student getting dressed for a winter school day. Which groups of tasks are sequential? Which groups of tasks are parallel?

put on socks
put on T-shirt
put on jeans
put on sweater
put on coat
put on toque
put on boxer shorts
put on watch
put on shoes
put on belt
put on scarf
put on gloves

**11.** **Application** Using the list of tasks from Question 10, create a task table that specifies which tasks are dependent on which tasks.

**12.** **Thinking, Inquiry, Problem Solving** Use the task table from Question 11 and create a project graph.

**13.** **Communication** What decisions need to be made before a critical path can be determined? (**Hint**: Does it matter what series of tasks get performed first?) Choose a critical path and explain why you think it is the optimal way to get dressed.

## Chapter Problem

### How Much Time Will It Take to Build This House?

**CP4.** The critical path for the Chapter Problem was provided as a solved example in this section. How would the time required to complete the project change if the contractor could convince the painter to complete that job in 11 d instead of 14 as originally scheduled? Explain.