6.4 Matrix Multiplication

Imagine that a builder constructs four different house models in three different cities. The following table shows the number of each model being built in each community.

	Deerfield	Cambridge	Exeter	Lansdowne
Oshawa	11	4	1	3
Milton	6	9	3	4
Port Credit	5	5	5	0

Each model has different design requirements. The number of windows and exterior doors required for each model is summarized below.

	Windows	Doors
Deerfield	16	3
Cambridge	20	4
Exeter	20	4
Lansdowne	12	2

How many windows and doors does the contractor need to order for all the houses that are to be built in each community? First, examine the requirements for the homes to be built in Oshawa. The total number of windows and doors required can be found by calculating the sum of the number of windows and doors required for each model of house being built. The calculations appear below.

Model	Windows	Doors
Deerfield	$11 \times 16 = 176$	$11 \times 3 = 33$
Cambridge	$4 \times 20 = 80$	$4 \times 4 = 16$
Exeter	$1 \times 20 = 20$	$1 \times 4 = 4$
Lansdowne	$3 \times 12 = 36$	$3 \times 2 = 6$
TOTAL	312	59

For the homes in Oshawa, the contractor must order 312 windows and 59 doors. This process would need to be repeated for the homes in Milton and Port Credit. To perform the calculations this way would be rather tedious. Instead, you can simplify it with the use of matrices.

Example 1 Using Matrix Multiplication

Recall that a matrix is a rectangular array of numbers with the row and column headings removed. The tables used to display the information on the previous page can each be written as a matrix. The matrix that provides the number of house models built in each city is a 3-by-4 matrix.

	Deerfield	Cambridge	Exeter	Lansdowne
Oshawa	11	4	1	3
Milton	6	9	3	4
Port Credit	5	5	5	0

$$C = \begin{bmatrix} 11 & 4 & 1 & 3 \\ 6 & 9 & 3 & 4 \\ 5 & 5 & 5 & 0 \end{bmatrix}$$

The information for the window and door requirements for each model can be placed in a 4-by-2 matrix.

	Windows	Doors
Deerfield	16	3
Cambridge	20	4
Exeter	20	4
Lansdowne	12	2

$$W = \begin{bmatrix} 16 & 3 \\ 20 & 4 \\ 20 & 4 \\ 12 & 2 \end{bmatrix}$$

The results can then be placed in a 3-by-2 matrix.

$$R = \begin{bmatrix} 312 & 59 \\ ? & ? \\ ? & ? \end{bmatrix}$$

Think about
Matrix Multiplication

- How were the entries in matrices *C* and *W* combined to find the total number of windows for Oshawa?
- How were they combined to find the total number of doors for Oshawa?

Product of Matrices

Matrix *R* is called the product of matrices *C* and *W*. The mathematical symbolism for this is $R = C \times W$.

Each entry in matrix *R* is calculated using the entries in the other two matrices. The process used to calculate the number of windows and the number of doors for the house to be built in Oshawa will be used to calculate the remainder of the entries.

Recall that the number of windows is $11 \times 16 + 4 \times 20 + 1 \times 20 + 3 \times 12 = 312$. Compare the entries in matrix *R* to the product of matrices *C* and *D*. This shows how the calculation was done.

$$\text{Compare } R = \begin{bmatrix} 312 & 59 \\ ? & ? \\ ? & ? \end{bmatrix} \text{ to } \begin{bmatrix} 11 & 4 & 1 & 3 \\ 6 & 9 & 3 & 4 \\ 5 & 5 & 5 & 0 \end{bmatrix} \times \begin{bmatrix} 16 & 3 \\ 20 & 4 \\ 20 & 4 \\ 12 & 2 \end{bmatrix}.$$

Symbolically, you have the following situation:

$$\begin{bmatrix} r_{1\,1} & r_{1\,2} \\ r_{2\,1} & r_{2\,2} \\ r_{3\,1} & r_{3\,2} \end{bmatrix} = \begin{bmatrix} c_{1\,1} & c_{1\,2} & c_{1\,3} & c_{1\,4} \\ c_{2\,1} & c_{2\,2} & c_{2\,3} & c_{2\,4} \\ c_{3\,1} & c_{3\,2} & c_{3\,3} & c_{3\,4} \end{bmatrix} \times \begin{bmatrix} w_{1\,1} & w_{1\,2} \\ w_{2\,1} & w_{2\,2} \\ w_{3\,1} & w_{3\,2} \\ w_{4\,1} & w_{4\,2} \end{bmatrix}$$

Think about
Matrix Multiplication
Why must the number of entries of each row of matrix *C* be equal to the number of entries of each column of matrix *W*?

The number of windows required for Oshawa is the entry $r_{1\,1}$ and it was calculated as follows:

$$r_{1\,1} = c_{1\,1}\,w_{1\,1} + c_{1\,2}\,w_{2\,1} + c_{1\,3}\,w_{3\,1} + c_{1\,4}\,w_{4\,1}$$

The number of doors required for Oshawa is the row entry $r_{1\,2}$ and it was calculated as $r_{1\,2} = c_{1\,1}\,w_{1\,2} + c_{1\,2}\,w_{2\,2} + c_{1\,3}\,w_{3\,2} + c_{1\,4}\,w_{4\,2}$.

$$\begin{bmatrix} 312 & 59 \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} 11 & 4 & 1 & 3 \\ 6 & 9 & 3 & 4 \\ 5 & 5 & 5 & 0 \end{bmatrix} \times \begin{bmatrix} 16 & 3 \\ 20 & 4 \\ 20 & 4 \\ 12 & 2 \end{bmatrix}$$

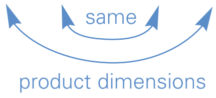
inner product—the number that is the sum of the products of each row entry from one matrix with its corresponding column entry from the other matrix.

One entry in the product matrix *R* is the **inner product** of a row from matrix *C* and the corresponding column from matrix *W*.

Matrix Dimensions

In general, if a matrix *A* with *m* rows and *k* columns is multiplied by a matrix *B* with *k* rows and *n* columns, the product will be a matrix *P* with *m* rows and *n* columns.

$$\begin{array}{ccccc} \boldsymbol{A} & \times & \boldsymbol{B} & = & \boldsymbol{P} \\ m \times k & & k \times n & & m \times n \end{array}$$



The remaining elements of the product matrix can be found using similar computations. For example, the total number of doors required for the house to be built in Port Credit is calculated using the following inner product:

Think about
Matrix Multiplication
Why can the matrix product *CW* (*C* × *W*) be calculated, but not the product *WC* (*W* × *C*)?

$$\begin{bmatrix} 11 & 4 & 1 & 3 \\ 6 & 9 & 3 & 4 \\ 5 & 5 & 5 & 0 \end{bmatrix} \times \begin{bmatrix} 16 & 3 \\ 20 & 4 \\ 20 & 4 \\ 12 & 2 \end{bmatrix} = \begin{bmatrix} 312 & 59 \\ ? & ? \\ ? & ? \end{bmatrix}$$

The value of $r_{3\,2}$ is the result of finding the inner product of row 3 of matrix *C* with column 2 of matrix *W*.

Example 2 Using a TI-83 Plus Calculator

Calculate the number of doors and windows required using matrix multiplication on a TI-83 Plus calculator.

Solution

To edit matrices, press 2nd MATRIX and use the arrow keys to scroll over to the EDIT menu. Press 1 to edit [A].

- Change the dimensions of [A] to 3 × 4.
- Enter the values one at a time, pressing ENTER after each. To view the entire matrix, you can use the arrow keys to scroll left or right, up or down.

```
MATRIX[A] 3 ×4
[ 0     0     0
[ 0     0     0
[ 0     0     0
1,1=0
```

```
MATRIX[A] 3 ×4
[ 11    4     1
3,1=5
```

Technolink
For more information on using a TI-83 Plus calculator to work with matrices, see Appendix C.15 and C.16 on pages 411 and 412.

- Use a similar sequence of steps to enter the window/door requirement information from matrix *W* into matrix [B].

```
MATRIX[B] 4 ×2
[ 16    3     ]
[ 20    4     ]
[ 20    5     ]
[ 12    2     ]
4,2=2
```

You may now return to the home screen and carry out the product. Press 2nd MATRIX 1 and 2nd MATRIX 2 to make [A] and [B] appear on the screen.

```
[A]*[B]
        [[312 59]
         [384 74]
         [280 55]]
```

The final results are as follows:

City	Windows	Doors
Oshawa	312	59
Milton	384	74
Port Credit	280	55

MATRICES AND GRAPHS

The Thamesville Maize, home of Ken and Ingrid Dieleman, **www.cornfieldmaze.com**. Photo taken by Austin Wright.

This aerial photograph shows a cornfield maze constructed in Thamesville, Ontario. A person enters the maze and tries to find a route that leads from the entrance to the exit. Because walls prevent the traveller from seeing the entire maze, the ability to make direction decisions at the end of each path intersection is limited.

INVESTIGATION: ESCAPING A MAZE

Examine the following maze. Is it possible to find a route through the maze that requires no backtracking?

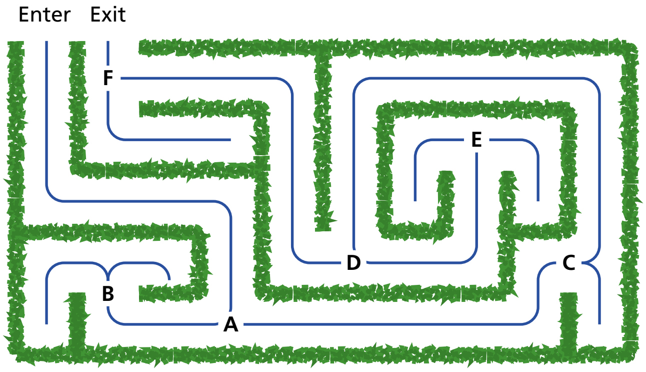
A. Copy the maze into your notebook. Label the places in the maze where a traveller would have to make a direction decision. Use the letters A, B, C, and so on. How many direction choices are there at each decision point?

B. Think of each decision point as a gate. The route through the maze consists of a sequence of paths that lead from one gate to the next. How many of these gate-to-gate paths are required for the shortest route through the maze?

REPRESENTING PATHS USING GRAPHS AND MATRICES

The individual pathways that lead from one point to the next, along with the points themselves, form a graph or network. The graph showing all possible routes in the maze from the Investigation is shown below.

Think about
Mazes
Graphs and networks were introduced in Section 6.1. How is the path through the maze like a graph?



A table can be used to show the number of one-step paths that link the lettered gates in this maze. These are called one-step paths because they join a gate to the next with only a single link.

In the table below

- the enter point, the lettered decision points, and the exit are shown down the side, and across the top.
- each cell in the table contains a 1 or a 0.
- a 1 shows that a link can be made directly from one point to the next. For example, if you read across the table from B, there is a "1" in column A. This means there is a link that goes from B to A.
- a 0 shows that there is no link.

TO

FROM

	Enter	A	B	C	D	E	F	Exit
Enter	0	1	0	0	0	0	0	0
A	1	0	1	1	0	0	0	0
B	0	1	0	0	0	0	0	0
C	0	1	0	0	1	0	0	0
D	0	0	0	1	0	1	1	0
E	0	0	0	0	1	0	0	0
F	0	0	0	0	1	0	0	1
Exit	0	0	0	0	0	0	1	0

Think About
Tables
Why does the table show that there is a path from A to B and also a path from B to A?

transition matrix—a matrix that shows the number of edges that connect the vertices of a graph

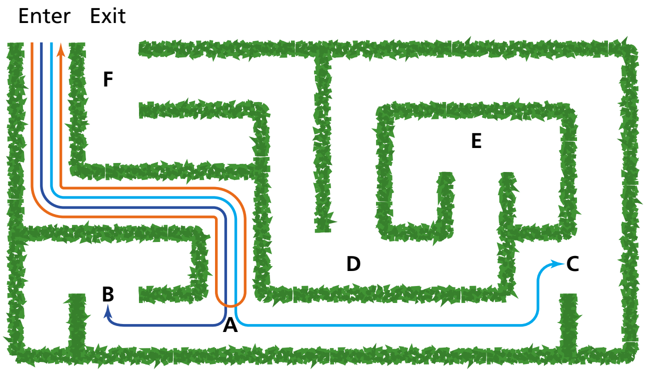
The information contained in this table can be modelled using the 8-by-8 **transition matrix** A shown below. Each row corresponds to a gate in the maze. Each column indicates whether or not a path exists directly from one gate to another.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

ANALYZING ROUTES USING A TRANSITION MATRIX

The transition matrix A (above) shows the number of paths joining vertices of the graph that are one edge in length. Now, consider paths that are two edges in length.

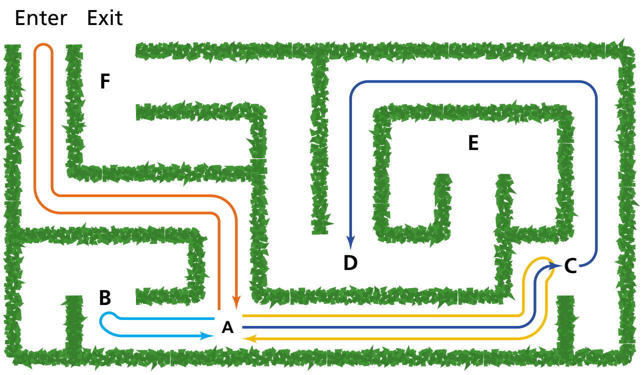
For example, there is a two-step path from Enter to B that passes through A, and also one from Enter to C. There is also a two-step path that goes from Enter to A and back to Enter. There are no two-step paths from Enter to A because the line must first pass through A.



Suppose you begin at A. There are three two-step paths that begin at A and end at A:

- A to Enter to A
- A to B to A
- A to C to A

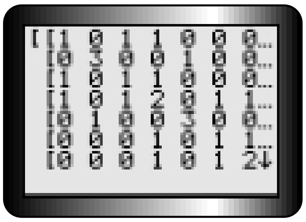
There is also a two-step path from A to D through C, but no others.



The partially completed transition matrix for two-step paths would have the following entries:

TO (in two steps)

		Enter	A	B	C	D	E	F	Exit
FROM	**Enter**	1	0	1	1	0	0	0	0
	A	0	3	0	0	1	0	0	0
	B								
	C								



```
[[1 0 1 1 0 0 0…
 [0 3 0 0 1 0 0…
 [1 0 1 1 0 0 0…
 [1 0 1 2 0 1 1…
 [0 1 0 0 3 0 0…
 [0 0 0 1 0 1 1…
 [0 0 0 1 0 1 2↓
```

Computing the remaining entries would be time-consuming. However, this two-step transition matrix turns out to be the product matrix $A^2 = A \times A$. Once the matrix entries have been entered into a TI-83 Plus calculator, the computation requires just pressing [A]*[A] or $[A]^2$. Use the arrow keys to scroll through the other entries in the matrix.

The completed two-step transition matrix is shown below.

TO (in two steps)

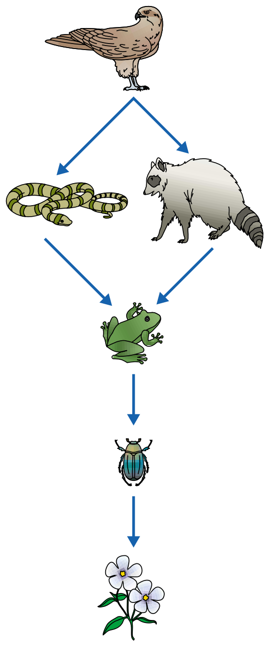
		Enter	A	B	C	D	E	F	Exit
FROM	**Enter**	1	0	1	1	0	0	0	0
	A	0	3	0	0	1	0	0	0
	B	1	0	1	1	0	0	0	0
	C	1	0	1	2	0	1	1	0
	D	0	1	0	0	3	0	0	1
	E	0	0	0	1	0	1	1	0
	F	0	0	0	1	0	1	2	0
	Exit	0	0	0	0	1	0	0	1

Think about
Transition Matrices

For what value of n does A^n show that an n-step route exists going from Enter to Exit? What does this mean?

USING MATRICES TO ANALYZE A FOOD CHAIN

A Food Chain



Chemical contaminants in the environment are passed up the food chain. For example, when the insecticide DDT is used, insects ingest it and pass it up the food chain to frogs and toads, who feed on the insects. It is then passed up the chain to snakes and racoons, who eat the toads and frogs. This continues until the animal at the top of the chain (often humans) is reached.

The simple food chain in the margin illustrates how a food chain can be thought of as a digraph in which the direction of the edge means “preys upon.” In table form, this becomes the following:

	hawk	snake	raccoon	frog	beetle	plant
hawk		Yes	Yes			
snake				Yes		
raccoon				Yes		
frog					Yes	
beetle						Yes
plant						

The predator–prey relationship in this diagram can be represented using a matrix. For example, the 1s in $m_{1\,2}$ and $m_{1\,3}$ represent the fact that the hawk preys upon the snake and the raccoon. The 1 in $m_{3\,4}$ indicates that the raccoon preys upon the frog.

$$M = \begin{bmatrix} 0 & \textcircled{1} & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix can be interpreted in a manner similar to the way a transition matrix models a maze. Each entry indicates whether or not a one-step path (or direct connection) exists between a predator and its prey. Each step represents the passing of DDT contamination along the food chain to a predator.

By examining the various powers of M, you can determine the number of steps required before the contamination reaches the top predator in the chain.

$$M^2 = \begin{bmatrix} 0 & 0 & 0 & \textcircled{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The 2 in row 1, column 4 of M^2 indicates that the hawk has indirectly fed on the frog by consuming one of two frog predators: the snake or the raccoon.

$$M^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \textcircled{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The 2 in row 1, column 5 of M^3 indicates that the hawk has indirectly fed on the bug via the frog and then the snake or the raccoon.

$$M^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The hawk will receive the DDT contamination when a non-zero entry appears in $m_{1\,6}$ since that indicates a path connecting the hawk to the plant. This occurs in M^4, indicating that four steps are required before the contamination reaches the top of the food chain.

KEY IDEAS

Product Matrix

- The product $P = A \times B$ of two matrices A and B is the matrix for which each entry $p_{r\,c}$ is the inner product of row r from matrix A with column c of matrix B.
- For the product to exist, the matrices must be compatible; that is, the number of columns of matrix A must equal the number of rows of matrix B.

Transition Matrix

- A transition matrix represents the number of edges that connect the vertices of a directed graph.

- Transition matrices may be used to represent paths through a maze, predator–prey relationships, or any other interrelationships that can be represented by a digraph.
- If T is a transition matrix for a digraph, then T^n shows all paths of length n between the vertices of the graph.

6.4 Exercises

A

1. Consider the following matrices.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad B = \begin{bmatrix} 10 & 20 & 30 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}$$

(a) Why is the matrix product BC possible, but the product CB is not?
(b) Find the dimensions of BA.
(c) Evaluate BA. Show your work.
(d) Evaluate the product AC or BC, whichever is possible.

2. If X is a 5-by-3 matrix and Y is a 3-by-5 matrix, what dimensions will the products XY and YX have? Explain your answer.

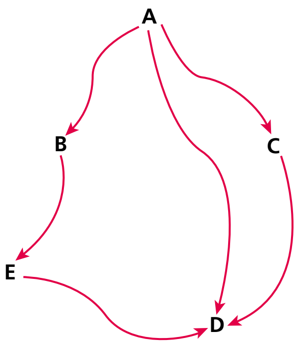
3. Perform the following matrix multiplications.

(a) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$

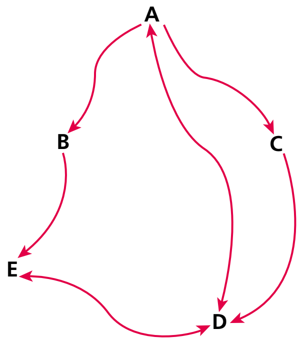
(b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$

4. **Knowledge and Understanding** Write the transition matrix that represents each of the following digraphs.

(a)



(b)



5. Draw the digraph that corresponds to this transition matrix.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

B

6. **Knowledge and Understanding**

For $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$,

(a) show that $(A \times B) \times C = A \times (B \times C)$
(b) show that $A \times B \neq B \times A$
(c) show that $(A - B) \times C = A \times C - B \times C$
(d) show that $(A + B) \times C = A \times C + B \times C$

7. For $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

(a) show that $(A \times B) \times C = A \times (B \times C)$
(b) show that $A \times B \neq B \times A$
(c) show that $(A - B) \times C = A \times C - B \times C$
(d) show that $(A + B) \times C = A \times C + B \times C$

8. Show that $AB = AC$.

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 5 & 6 \\ -1 & -3 \end{bmatrix}$$

9. **Thinking, Inquiry, Problem Solving** Find examples of two 2-by-2 matrices A and B for which $AB = BA$.

10. A bicycle manufacturer has tracked the time required for each stage of the manufacturing process for each of two models. The times required for each bike model are shown below.

Model	Assembly Time (min)	Painting Time (min)	Packing Time (min)
Kiddie	20	5	10
Adult	30	10	10

Assemblers earn \$20/h, painters earn \$15/h, and packers earn \$10/h. Use matrix methods to determine the total labour cost associated with each bike.

11. **Application** To make the manufacturing process efficient, a window supplier has designed windows so that each style can be made using the same-sized frame pieces and the same-sized glass sections. The requirements for each style are listed below.

Style	Frame Pieces	Glass Sections
A	4	1
B	7	2
C	13	4
D	16	5

A builder has several different models of homes for which she has ordered windows. The requirements for each model are listed in the following table.

	Window Styles			
Model	A	B	C	D
Descartes	5	2	0	1
Gauss	4	4	2	0
Fermat	4	1	3	2

Find the total number of frame pieces and glass sections the manufacturer will need to use for each model.

12. Bombay Pacific airlines charges different amounts based on the class of ticket purchased. For a flight from Toronto to Vancouver, the price schedule is \$750 for first class, \$625 for business class, and \$500 for economy. In one week of travel, the number of tickets sold for each class is shown in the table below. Use matrix methods to calculate the daily total revenue for the airline.

	First Class	Business Class	Economy
Monday	10	15	130
Tuesday	12	17	140
Wednesday	11	20	150
Thursday	15	15	200
Friday	20	11	210
Saturday	8	0	220
Sunday	2	0	165

13. The window supplier described in Question 11 knows that each frame piece costs $2.00 and each glass section costs $15.00.

(a) Use matrix methods to determine the cost of each window.

(b) Use matrix methods to determine the total cost to the window supplier for the windows needed for each model of home.

14. The following chart indicates the routes flown by a small airline.

From/To	Calgary	Hamilton	Edmonton	Regina	Ottawa
Calgary		N	Y	Y	Y
Hamilton	N		N	N	Y
Edmonton	Y	N		N	Y
Regina	N	N	N		Y
Ottawa	Y	Y	Y	Y	

(a) Use matrix methods to determine if a passenger can reach Hamilton in a round-trip flight from Calgary.

(b) It is not possible for a passenger to fly directly from Edmonton to Regina. Use matrix methods to determine if it is possible to do so using two flights.

(c) How many different routes are possible for a passenger to fly from Ottawa and return using three flights? Show your work.

C **15.** Five friends communicate by telephone. However, not all of them will speak to one another directly.

- Della will phone Billy, Ahmed, and Dahlia, but not David.
- Billy will phone only Ahmed and Della.
- Dahlia will phone only David and Della.
- David will phone only Billy and Dahlia.
- Ahmed won't phone anybody.

(a) How many calls are needed for Billy to send a message to David?

(b) Show whether it is possible for Dahlia to send a message and get it back in three phone calls.

16. A baker is planning her next day's production requirements. She knows that a loaf of cottage bread requires 1.5 L of flour, two packages of yeast, and 100 mL of oil. Each loaf of white bread requires 2 L of flour, two packages of yeast, and 75 mL of oil. Each loaf of buttermilk bread requires 1.25 L of flour, two packages of yeast, and 60 mL of oil.

Flour costs $1.25 per litre, oil costs $0.30 per 100 mL, and yeast costs $0.25 a package.

(a) Use matrix methods to compute the cost of the ingredients for each type of bread.

(b) Use matrix methods to determine the cost of the ingredients for an order consisting of 30 cottage loaves, 120 white loaves, and 50 loaves of buttermilk bread.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

In Section 6.2 you were introduced to a friendship matrix that led to a graph of student relationships.

Would work with ...	Barb	Amir	Ken	Rene	Sean
Barb		+	+	−	+
Amir	−		+	+	−
Ken	+	−		−	−
Rene	+	+	+		+
Sean	+	−	+	+	

17. **Knowledge and Understanding** Write this chart as matrix F in which "would work with" is represented by a 1 and "would not work with" is represented by a 0. Assume that students would work with themselves.

18. **Application** Compute F^2 and F^3.

19. **Thinking, Inquiry, Problem Solving** Imagine that the five students were asked to rate the others as follows: most-wanted partner (1) to least-wanted partner (5). Create a way of recording this result in a matrix R. What does R^2 mean?

20. **Communication** What information about the working relationships among students does F^2 provide?