6.3 Organizing Information with Matrices

The following table shows the breakdown by grade of student enrolment in Ontario schools in the school year 1998–99.

Grade/Age	Public Schools	Separate Schools
JK	66 113	43 734
K	91 149	47 813
Grade 1	96 878	49 834
Grade 2	96 618	48 832
Grade 3	97 185	49 182
Grade 4	94 650	47 169
Grade 5	91 885	45 150
Grade 6	91 812	44 482
Grade 7	93 309	44 403
Grade 8	92 778	44 336
Ungraded (elementary)	1 274	117
Special Education (elementary)	29 755	5 328
Pre-Grade 9 (secondary)	1 444	298
Grade 9	110 437	44 131
Grade 10	113 379	41 939
Grade 11	107 250	40 296
Grade 12	162 147	64 202
Age 21+	11 139	1 174

Source: Web site of Ontario Ministry of Education

DEFINITION OF A MATRIX

The meaning of a value in the table depends on its row and column position. For example, the number circled in red in the table above tells you there were 110 437 Grade 9 students enrolled in public schools in 1998–99.

While the row and column headings are useful, the real information in the table consists of the numbers themselves. The numbers remaining after the row and column headings are removed form a rectangular array of numbers. Such an array is called a **matrix**.

matrix—a rectangular array of numbers set out in rows and columns

The matrix formed from the table appears to the right. A matrix is always bordered by square brackets to distinguish it from other rectangular arrays of information. Columns indicate public and separate schools.

For convenience, a matrix is labelled with a single capital letter, in this case, P. The lowercase letter p is used to refer to the individual entries in the matrix. Subscripts indicate the row and column position. For example, the number circled in red is in the fourteenth row and first column. Thus, $p_{14\,1} = 110\,437$. The number circled in blue is referred to as $p_{3\,2} = 49\,834$. It is in row 3, column 2.

Since this matrix has 18 rows and 2 columns, it is called an 18-by-2 matrix; these numbers are known as the dimensions of the matrix.

columns indicate public and separate schools

$$P = \begin{bmatrix} 66\,113 & 43\,734 \\ 91\,149 & 47\,813 \\ 96\,878 & 49\,834 \\ 96\,618 & 48\,832 \\ 97\,185 & 49\,182 \\ 94\,650 & 47\,169 \\ 91\,885 & 45\,150 \\ 91\,812 & 44\,482 \\ 93\,309 & 44\,403 \\ 92\,778 & 44\,336 \\ 1\,274 & 117 \\ 29\,755 & 5\,328 \\ 1\,444 & 298 \\ 110\,437 & 44\,131 \\ 113\,379 & 41\,939 \\ 107\,250 & 40\,296 \\ 162\,147 & 64\,202 \\ 11\,139 & 1\,174 \end{bmatrix}$$

rows indicate grade and age

Think about
Matrix Labels

The elements in the enrolment matrix would be labelled as follows:

$$P = \begin{bmatrix} p_{1\,1} & p_{1\,2} \\ p_{2\,1} & p_{2\,2} \\ p_{3\,1} & p_{3\,2} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ p_{17\,1} & p_{17\,2} \\ p_{18\,1} & p_{18\,2} \end{bmatrix}$$

Explain how the letters and numbers are used.

Matrix Terminology

In general, if a matrix A has m rows and n columns, it is called an m-by-n matrix. The variables m and n represent the dimensions of the matrix.

The entry, a, in row i and column j is represented as $a_{i\,j}$.

Example 1 Row and Column Operations

What was the total number of Ontario students in each grade or age category in 1998–99?

Solution 1 *No technology required*

Since each row in the matrix corresponds to one of the grade/age categories, each **row sum** will provide the total number of students enrolled at each grade/age level. Similarly, each **column sum** will provide the total enrolment for either the public or separate school systems.

row sum—the sum of the elements in one row of a matrix

column sum—the sum of the elements in one column of a matrix

Solution 2 *Using spreadsheet software*

For simple computations with large matrices, a spreadsheet is a useful tool. The spreadsheet that follows shows the formulas required to calculate the row sums and the column sums for the 1998–99 matrix of student enrolments.

Think about Spreadsheets
What formula would find the combined total of all students enrolled in both school systems for 1998–99?

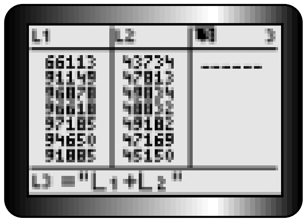
	A	B	C
1	66113	43734	=SUM(A1:B1)
2	91149	47813	=SUM(A2:B2)
3	96878	49834	=SUM(A3:B3)
4	96618	48832	=SUM(A4:B4)
5	97185	49182	=SUM(A5:B5)
6	94650	47169	=SUM(A6:B6)
7	91885	45150	=SUM(A7:B7)
8	91812	44482	=SUM(A8:B8)
9	93309	44403	=SUM(A9:B9)
10	92778	44336	=SUM(A10:B10)
11	1274	117	=SUM(A11:B11)
12	29755	5328	=SUM(A12:B12)
13	1444	298	=SUM(A13:B13)
14	110437	44131	=SUM(A14:B14)
15	113379	41939	=SUM(A15:B15)
16	107250	40296	=SUM(A16:B16)
17	162147	64202	=SUM(A17:B17)
18	11139	1174	=SUM(A18:B18)
19	=SUM(A1:A18)	=SUM(B1:B18)	

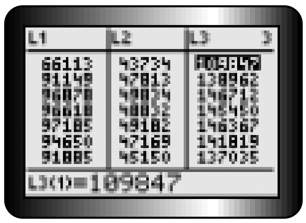
Technolink
For more information on using formulas in spreadsheets, see Appendix E.1 on page 425.

Solution 3 *Using a TI-83 Plus calculator*

Use the following procedure to find the row sums:

- Enter the values from the matrix columns into $\mathbf{L_1}$ and $\mathbf{L_2}$ using the **STAT EDIT** command.
- Use the arrow-up key to move the insertion point to the top of $\mathbf{L_3}$.
- Enter the formula $\mathbf{L_1} + \mathbf{L_2}$ into the top of $\mathbf{L_3}$. This tells the calculator to add the elements in each row and enter the sums in the same row of $\mathbf{L_3}$.





To find the column sums, you must return to the home screen and use the **sum** command. It is found by pressing [2nd] [LIST] and using the arrow keys to scroll over to the **MATH** menu.

Technolink
To update a column result, enclose the formula in L_3 in quotes (SHIFT+); otherwise, you will need to re-enter the formula to recalculate the column.

Example 2 **Multiplying by a Scalar**

Suppose the government spent an average of $1500 per student for education. Calculate the expenditure for each grade/age category and school system.

Solution 1 *No technology required*

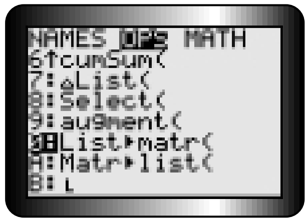
scalar—a quantity with magnitude but no direction

Multiply each entry in the enrolment matrix by 1500, a **scalar** quantity, to create a new matrix that shows the expenditures. Manually, for this matrix, this would be a very tedious operation.

$$\begin{bmatrix} 1500 \times 66\,113 & 1500 \times 43\,734 \\ 1500 \times 91\,149 & 1500 \times 47\,813 \\ 1500 \times 96\,878 & 1500 \times 49\,834 \\ \vdots & \vdots \end{bmatrix}$$

Technolink For more information on using a TI-83 Plus calculator to work with matrices, see Appendix C.15 on page 411.

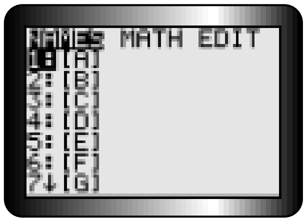
Solution 2 *Using a TI-83 Plus calculator*



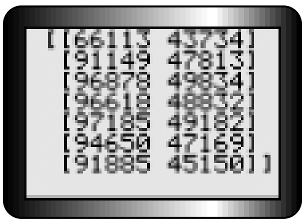
A TI-83 Plus calculator will multiply the elements of a matrix by a scalar. You can create a new matrix and enter the elements into it, or you can create a matrix from the elements in the two separate lists $\mathbf{L_1}$ and $\mathbf{L_2}$. This is done using the [2nd] [LIST] **OPS List▶ matr(** command.

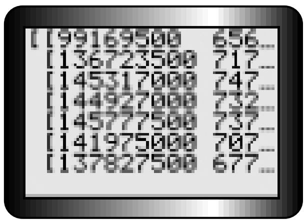
Immediately following the left parenthesis, enter the names, separated by commas, of the lists that will form the columns of the matrix.

To identify the matrix in which the results will be stored, use the **MATRIX NAMES** command. In this case, matrix **[A]** was selected.



The original matrix **[A]** and the matrix produced by entering the calculation **1500*[A]** appear below. The arrow keys are used to scroll vertically or horizontally so that all the entries can be viewed.





Technolink You may store the results of your matrix calculation in any of the matrices in the NAMES list by using the [STO▸] key.

The multiplication could have been entered as 1500[A]. The use of the * is not required.

Example 3 Subtracting Matrices

Use the data in the following tables to compute the changes in Ontario student enrolment from 1997–98 to 1998–99.

Student Enrolment in Ontario (1997–1998)		
	Elementary	Secondary
Public Schools		
Male	477 349	266 694
Female	448 599	246 679
Separate Schools		
Male	238 426	93 242
Female	230 065	94 576

Student Enrolment in Ontario (1998–1999)		
	Elementary	Secondary
Public Schools		
Male	486 350	263 190
Female	457 056	242 606
Separate Schools		
Male	239 104	95 159
Female	231 276	96 881

Solution

The changes in enrolment for each category in the table can be calculated by subtracting the values in the 1997–98 table from the corresponding values in the 1998–99 table. Matrices can be used to help you.

Create a matrix for each table. Matrix A contains the data for 1998–99 and matrix B contains the data for 1997–98.

$$A = \begin{bmatrix} 486\,350 & 263\,190 \\ 457\,056 & 242\,606 \\ 239\,104 & 95\,159 \\ 231\,276 & 96\,881 \end{bmatrix} \qquad B = \begin{bmatrix} 477\,349 & 266\,694 \\ 448\,599 & 246\,679 \\ 238\,426 & 93\,242 \\ 230\,065 & 94\,576 \end{bmatrix}$$

The enrolment changes can be found by creating a new matrix $C = A - B$. To subtract matrices, each element in the second matrix (matrix B) is subtracted from the corresponding element in the first matrix (matrix A) as shown below.

Think about Subtracting Matrices

Why must the matrices being subtracted have the same dimensions?

$$C = A - B$$

$$= \begin{bmatrix} 486\,350 & 263\,190 \\ 457\,056 & 242\,606 \\ 239\,104 & 95\,159 \\ 231\,276 & 96\,881 \end{bmatrix} - \begin{bmatrix} 477\,349 & 266\,694 \\ 448\,599 & 246\,679 \\ 238\,426 & 93\,242 \\ 230\,065 & 94\,576 \end{bmatrix}$$

$$= \begin{bmatrix} 486\,350 - 477\,349 & 263\,190 - 266\,694 \\ 457\,056 - 448\,599 & 242\,606 - 246\,679 \\ 239\,104 - 238\,426 & 95\,159 - 93\,242 \\ 231\,276 - 230\,065 & 96\,881 - 94\,576 \end{bmatrix}$$

$$= \begin{bmatrix} 9001 & -3504 \\ 8457 & -4073 \\ 678 & 1917 \\ 1211 & 2305 \end{bmatrix}$$

Technolink For more information on using a TI-83 Plus calculator to add, subtract, or multiply matrices, see Appendix C.16 on page 412.

The calculation can be carried out on a TI-83 Plus calculator.

The enrolment changes are displayed in the table to the right.

Changes in Ontario Student Enrolment from 1997–1998 to 1998–1999

	Elementary	Secondary
Public Schools		
Male	9001	–3504
Female	8457	–4073
Separate Schools		
Male	678	1917
Female	1211	2305

Sum and Difference of Matrices

If matrices A and B have the same dimensions, then

- the sum $S = A + B$ is the matrix formed by adding the entries of matrix A to the corresponding entries of matrix B

$$s_{ij} = a_{ij} + b_{ij}$$

- the difference $D = A - B$ is the matrix formed by subtracting the entries of matrix B from the corresponding entries of matrix A

$$d_{ij} = a_{ij} - b_{ij}$$

Project Connection

You may be gathering large amounts of information for your course project. Much of these data may be in the form of tables of numbers. Matrix operations using spreadsheets or calculators will make computations with these data easier.

KEY IDEAS

matrix—a rectangular array of numbers set out in rows and columns

- The entries or elements in the matrix are labelled according to the row and column in which each appears. For example, the element $a_{3\,7}$ is the value found in row 3 and column 7 of matrix A.
- The dimensions of a matrix correspond to the number of rows and the number of columns in the matrix. For example, a 3-by-5 matrix has 3 rows and 5 columns.

Simple Matrix Operations

- The entries in a row or column of a matrix may be added to form a row sum or a column sum.
- Two matrices of equal dimension may be added or subtracted by adding or subtracting the corresponding entries within each matrix.
- A matrix may be multiplied by a scalar. Each entry in the matrix is multiplied by the same number.

6.3 Exercises

A

1. Given matrices A and B below, find

$$A = \begin{bmatrix} 2 & 4 & 9 \\ 3 & 10 & 0 \\ 4 & 11 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 5 & 7 & 9 \\ 9 & 25 & 49 & 81 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -1 \\ 1 & 8 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 2 & 5 \\ 5 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 5 & 2 & -7 & 0 \\ 3 & 9 & 6 & 4 \end{bmatrix}$$

(a) $a_{2\,3}$ **(b)** $c_{3\,2}$ **(c)** $b_{1\,3}$ **(d)** $d_{2\,2}$
(e) the dimensions of matrix A and matrix B
(f) the row sums for the matrix A
(g) the column sums for the matrix B

2. **Knowledge and Understanding** Using the matrices from Question 1, determine the following if possible.
(a) $3A$ **(b)** $E \times 0.10$ **(c)** $A + C$ **(d)** $A - B$
(e) $D + E$ **(f)** $F - B$ **(g)** $2E - D$ **(h)** $5F - B$

3. **Communication** Explain why matrices A and B in Question 1 may not be added or subtracted, while matrices A and C may.

4. For matrix A in Question 1, determine the position in the form $a_{i\,j}$ of each of these numbers.
(a) 2 **(b)** 3 **(c)** 0 **(d)** 10 **(e)** 11

B

5. **Application** Use matrix operations to get career totals for Bobby Orr from the following regular season data. (G: goals, A: assists, Pts: points, PIM: penalties in minutes)

Year	Games	G	A	Pts	PIM
1966–67	61	13	28	41	102
1967–68	46	11	20	31	63
1968–69	67	21	43	64	133
1969–70	76	33	87	120	125
1970–71	78	37	102	139	91
1971–72	76	37	80	117	106
1972–73	63	29	72	101	99
1973–74	74	32	90	122	82
1974–75	80	46	89	135	101
1975–76	10	5	13	18	22
1976–77	20	4	19	23	25
1978–79	6	2	2	4	4

Source: Orr Fan (**http://www.orrfan.com/stats.htm**) © H. Holman

	Employment		
	Sept. '01	Oct. '01	% Change
Total	15 093.6	15 095.4	0.0
NF & L	212.0	212.9	0.4
PEI	66.0	66.0	0.0
NS	425.6	429.7	1.0
NB	333.8	336.6	0.8
PQ	3 497.1	3 501.7	0.1
ON	5 958.6	5 955.7	0.0
MB	558.7	562.9	0.8
SK	468.9	468.0	–0.2
AB	1 638.9	1 641.3	0.1
BC	1 934.1	1 920.6	–0.7

Source: Statistics Canada, *The Daily*, Nov. 2, 2001

6. **Knowledge and Understanding** In the table to the left, the first column contains the number of people employed in September 2001, the second column contains the data for October 2001, and the third column contains the percent increase from September to October.

(a) Write matrix E that would contain these data.

(b) Which number would be found in $e_{6\,1}$? Which province corresponds to this employment data?

(c) Which element $e_{j\,k}$ in matrix E would contain the October employment statistics for Alberta?

7. The following table lists the number of Canadians 15 years of age or older by their highest degree or certificate.

Definitions and notes	1986	1991	1996
Total	19 634 100	21 304 740	22 628 925
No degree, certificate, or diploma	9 384 100	8 639 900	8 331 615
Secondary (high) school graduation certificate	3 985 820	4 967 325	5 217 205
Trades certificate or diploma	1 989 850	2 342 105	2 372 000
Other non-university certificate or diploma	2 034 485	2 494 460	3 181 840
University certificate or diploma below bachelor level	381 580	441 205	525 560
Bachelor's degree	1 254 250	1 585 775	1 979 460
University certificate or diploma above bachelor level	189 000	264 845	310 820
Medical degree	74 945	90 835	105 050
Master's degree	293 335	394 750	501 505
Earned doctorate	66 955	83 545	103 855

Source: Statistics Canada, 1996 Census *Nation* tables

Use matrix operations to

(a) find the change from 1986 to 1991 in the number who earned a doctorate

(b) verify that the number in the Total row for each year is correct

(c) compute the total amount of money spent obtaining a Bachelor's degree in 1991 and 1996, if the average degree costs $50 000.

8. **Application** The following table summarizes the number of Canadians 15 years of age and older who were involved in a variety of sports activities in 1998.

Sports Activity	Males (1000s)	Females (1000s)
Golf	1325	476
Hockey (ice)	1435	65
Baseball	953	386
Swimming	432	688
Basketball	550	237
Volleyball	394	350
Soccer	550	189
Tennis	434	224
Skiing, downhill/alpine	342	315
Cycling	358	250
Skiing, cross-country/nordic	208	304
Weightlifting	294	140
Badminton	199	204
Football	347	40
Curling	179	133
Bowling, 10 pin	132	150
Softball	118	92
Bowling, 5 pin	78	122

Source: Statistics Canada

(a) Use matrix methods to calculate
  (i) the total number of Canadians who played each sport
  (ii) the total number of males involved in sports activities
  (iii) the total number of females involved in sports activities

(b) Canadians played sports other than those shown in the table above. The total number of males in the original data was 11 937 000 and the total number of females was 12 323 000. Use matrix methods to determine the percent of total males and the percent of total females involved in each of the sports activities reported above.

C **9.** Ever-Green Pine Trees has compiled the following sales data:

	1998	1999	2000	2001
Scotch Pine	58	44	51	39
Douglas Fir	13	15	19	22
Blue Spruce	25	30	22	24

(a) Using matrix methods, determine which year was the best year for sales.
(b) If Scotch Pine were sold for $25, Douglas Fir for $40, and Blue Spruce for $20, which year generated the most revenue?
(c) For every tree that is cut down, the company must plant three in its place. How many of each species of tree needs to be planted to replace the trees sold in the four years recorded above?

10. Consider the following results from the 2000 Summer Olympic Games in Sydney, Australia:

Country	Gold	Silver	Bronze
United States	39	25	33
Russia	32	28	28
China	28	16	15
Australia	16	25	17
Germany	14	17	26
France	13	14	11
Italy	13	8	13
The Netherlands	12	9	4
Cuba	11	11	7
Great Britain	11	10	7

Source: British Broadcasting Corporation (BBC)

(a) Record the results in matrix form.
(b) If a gold medal were worth 5 points, a silver worth 3 points, and a bronze worth 1 point, which country would receive the most points?
(c) Using the Internet, find the medal results for the top 10 countries from the last Olympic Games. Which country would receive the most points this time?

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

The following chart lists VGY Toy Company's inventory of their top-selling toys in their four local warehouses.

		Warehouses			
		N	E	S	W
Toy	Fizzie Ball	34	40	37	25
	Quaterno	15	22	10	13
	The Baffo Game	27	34	13	22

11. **Knowledge and Understanding** For quality control purposes, the company has decided to select 5% of their inventory for inspection. Using matrix multiplication, determine the number of each kind of toy at each location that is to be selected for inspection.

12. **Application** Using matrix methods, determine the number of toys at each warehouse, as well as the total number of each toy in all four warehouses.

13. **Thinking, Inquiry, Problem Solving** Head office has ordered that no more than 50% of the inventory of any warehouse is to be made up of Fizzie Balls. What exchanges of merchandise are necessary to comply with this directive?

14. **Communication** The following matrix contains the retail price in dollars of the Fizzie Ball, Quaterno, and The Baffo Game, respectively:

$$[17.99 \quad 27.95 \quad 22.88]$$

Describe how you would determine the value of the inventory in warehouse N.

Chapter Problem

How Much Time Will It Take to Build This House?

CP3. The contractor has determined the number of days and the number of workers required to complete each task in the construction of the house. Use the information in the table below to construct a matrix showing the number of days and workers.

(a) Use the matrix to determine the total number of person days required for each task, as well as the total number of person days needed to complete the project.

(b) If each worker is paid $15/h, estimate the total labour cost for the project.

Task	Description	Duration (days)	Workers
Start	Acquire land	0	2
A	Stake lines and grades	5	2
B	Clear site of trees, etc., and do rough grading	5	4
C	Install service for all utilities	7	3
D	Excavate	6	3
E	Lay foundation, pour basement, and then backfill	8	5
F	Frame doors, walls, and roof	19	6
G	Shingle roof	7	2
H	Install doors and windows	5	2
I	Build exterior wall surfaces	13	3
J	Rough-in ductwork, plumbing, and electrical	17	3
K	Install drywall	20	2
L	Paint	11	1
M	Do finish carpentry	14	1
N	Do plumbing, heating, and electrical finish work	9	3
O	Install flooring	9	2
P	Complete legal work and transfer of ownership	7	1
Finish			