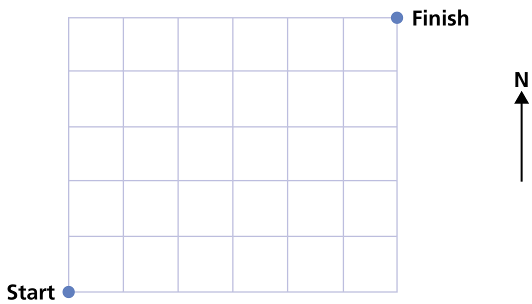
5.2 Pascal's Triangle and the Binomial Theorem

What does computing the expansion of an algebraic expression such as $(a + b)^4$ have in common with finding the number of pathways through a city's street system? It turns out that both depend on calculating combinations. In fact, the two problems are similar.

Example 1 Paths Through a Map Grid

The streets of a city are laid out in a rectangular grid, as shown. By travelling either east or north along the streets, how many possible routes lead from start to finish?



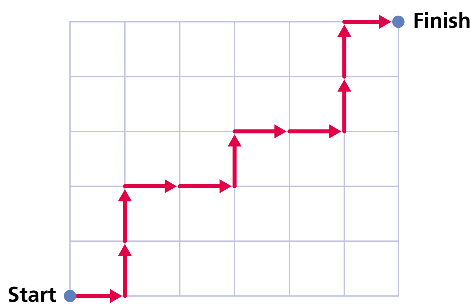
Solution

You must move 6 streets east and 5 streets north to get from start to finish. Therefore, every possible route requires 11 steps. Of these steps, 5 will move you north and 6 will move you east. You can record a route by filling in 11 spaces with 6 E's (east) and 5 N's (north) in a route record. An example of such a route record, as well as a diagram of the corresponding route map, appears below.

Think about The Route Record

Why does knowing the number of E's in the record automatically tell you the number of N's?

E	N	N	E	E	N	E	E	N	N	E



Calculating the number of routes is equivalent to determining the number of ways E can be inserted into 6 positions selected from the 11 available positions. This can be done using combinations in $C(11, 6)$ or $\binom{11}{6} = 462$ ways.

Think about Counting the Placement of E's

Why does it not matter whether you count the number of ways the E's can be placed or the ways the N's can be placed?

BINOMIAL THEOREM

The expansions for several different powers of $(a + b)$ are listed below. The French mathematician Blaise Pascal noted a pattern in the expansion of these powers. As a result, the triangular array of coefficients of these expansions became known as *Pascal's Triangle*.

Think about **Pascal's Triangle**

The first five rows are

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Determine the sixth and seventh rows.

$$(a + b)^0 = 1$$
$$(a + b)^1 = a + b$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

The values of the coefficients in the expansions can be determined using a strategy very similar to the one used previously to analyze paths through a map grid. For example, consider the term that includes a^3b in the expansion of $(a + b)^4$. It is the result of multiplying the a-term from three of the factors with the b-term from the remaining factor. The product, *aaab*, is illustrated below.

$$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$$

a	*a*	*a*	*b*
a	*a*	*b*	*a*
a	*b*	*a*	*a*
b	*a*	*a*	*a*

The four ways that this can be done appear here.

There are four available places to record a's and b's. For the term a^3b, one space has to be occupied by a b. Therefore, there are $C(4, 1)$ ways of doing this. Hence, the coefficient of the a^3b term is $C(4, 1) = \binom{4}{1}$ or 4. As a result, the coefficients of the general binomial expansion $(a + b)^n$ can be represented by Pascal's Triangle in terms of the combination formula $\binom{n}{r}$. The numerical values appear to the right.

Think about **The Coefficients of a^3b**

Does it matter whether you count the ways the three a's can be placed or the ways the one b can be placed?

Pascal's Triangle Using $\binom{n}{r}$

Value of *n*	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
0	$\binom{0}{0}$						
1	$\binom{1}{0}$	$\binom{1}{1}$					
2	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{2}{2}$				
3	$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	$\binom{3}{3}$			
4	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$		
5	$\binom{5}{0}$	$\binom{5}{1}$	$\binom{5}{2}$	$\binom{5}{3}$	$\binom{5}{4}$	$\binom{5}{5}$	
6	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{6}{2}$	$\binom{6}{3}$	$\binom{6}{4}$	$\binom{6}{5}$	$\binom{6}{6}$

Pascal's Triangle Numerically

Value of *n*							
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

In both representations, n represents a row in the triangle as well as the exponent in the expansion of $(a + b)^n$. We can generalize the results of the expansions above $[(a + b)^n$, where $n = 0, 1, 2, \ldots, 5]$ and Pascal's Triangle with the Binomial Theorem, as stated on the following page.

Think about $\binom{n}{r}a^{n-r}b^r$

If n is the exponent in $(a+b)^n$, what does r represent?

Binomial Theorem

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + \binom{n}{n}b^n$$

The coefficients of the form $\binom{n}{r}$ are called binomial coefficients.

Example 2 Using the Binomial Theorem

Expand using the Binomial Theorem.

(a) $(a+b)^8$ **(b)** $(2x-3)^5$

Solution

(a)

$$(a+b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 + \binom{8}{4}a^4b^4 + \binom{8}{5}a^3b^5 + \binom{8}{6}a^2b^6 + \binom{8}{7}ab^7 + \binom{8}{8}b^8$$

$$= a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$$

(b) In $(2x-3)^5$, let $n = 5$, $a = 2x$, and $b = -3$.

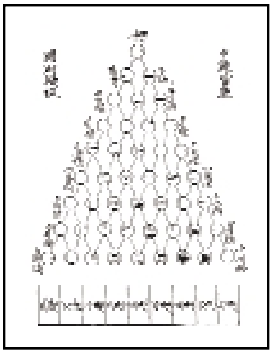
Therefore,

$$(2x-3)^5 = \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(-3) + \binom{5}{2}(2x)^3(-3)^2 + \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)(-3)^4 + \binom{5}{5}(-3)^5$$

$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243$$

Think about **Pascal's Triangle**

A version of Pascal's Triangle was known to the Chinese long before the birth of Blaise Pascal. This diagram comes from Chu Shih-Chieh's *Precious Mirror of the Four Elements,* published in 1303.



INVESTIGATION 2: BINOMIAL THEOREM

Numerous patterns can be found in the expansion of a binomial.

Purpose

To identify patterns in the binomial coefficients using the Binomial Theorem.

Procedure

A. Write the expansion of $(x+y)^6$ using the Binomial Theorem.
B. Write the expansion of $(x+y)^9$ using the Binomial Theorem.

Discussion Questions

1. Using the expansion of $(x+y)^6$, answer the questions that follow.
  (a) In the term containing x^5, what is the exponent of the y-term? What is the coefficient of this term?

(b) Why is the value of the coefficient of the term containing x^5 the same as the coefficient of the term containing y^5?

(c) How is finding the coefficient in part (a) similar to finding routes through a street grid following a five-east, one-north path?

2. Using the expansion of $(x + y)^9$, answer the questions that follow.

(a) In the term containing x^5, what is the exponent of the y-term? What is the coefficient of this term?

(b) Why is the value of the coefficient of the term containing x^5 the same as the coefficient of the term containing y^5?

Think about The Exponents over *a* and *b*

If the exponent over a is r, why is the exponent over b always $n - r$?

PASCAL'S TRIANGLE

Blaise Pascal discovered many interesting patterns in the coefficients in the expansions of $(a + b)^n$. Some of the patterns become easier to see when the triangle is arranged as it is to the right.

As discussed previously, the entry in the $n = 4$ row and $r = 3$ column corresponds to the coefficient of a^3b in the expansion of $(a + b)^4$ and has a value of $C(4, 1)$, or 4.

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
$n = 0$	1						
$n = 1$	1	1					
$n = 2$	1	2	1				
$n = 3$	1	3	3	1			
$n = 4$	1	**4**	6	4	1		
$n = 5$	1	5	10	10	5	1	
$n = 6$	1	6	15	20	15	6	1

Perhaps the most famous pattern in Pascal's Triangle stems from the relationship between the sum of consecutive values in one row and the value found in the next row immediately beneath.

The diagram suggests that $\binom{4}{1} + \binom{4}{2} = \binom{5}{2}$. This can be verified by direct computation.

$$\begin{aligned}\binom{4}{1} + \binom{4}{2} &= \frac{4!}{1!\,3!} + \frac{4!}{2!\,2!}\\ &= 4 + 6\\ &= 10\end{aligned}$$

$$\begin{aligned}\binom{5}{2} &= \frac{5!}{2!\,3!}\\ &= 10\end{aligned}$$

	$r = 0$	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
$n = 0$	1						
$n = 1$	1	1					
$n = 2$	1	2	1				
$n = 3$	1	3	3	1			
$n = 4$	1	**4**	**6**	4	1		
$n = 5$	1	5	**10**	10	5	1	
$n = 6$	1	6	15	20	15	6	1

Think about The Steps in the Proof

- Where in the proof is a common denominator used to add fractions?
- How is the fact that for any value of k, $k!(k + 1) = (k + 1)!$ used in the proof?
- Where is common factoring used to simplify an expression in the proof?

Pascal's Identity

As a result of the importance of this particular relationship in Pascal's Triangle, it has become known as Pascal's Identity. An identity is a relationship that is true for all possible values of the variables involved.

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

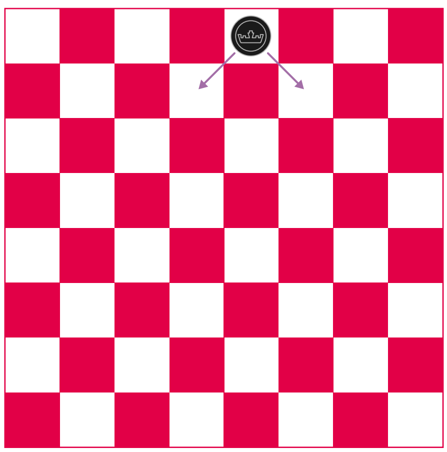
Proof

$$\begin{aligned}
&\binom{n}{r} + \binom{n}{r+1} \\
&= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)![n-(r+1)]!} \\
&= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{(r+1)n!}{(r+1)r!(n-r)!} + \frac{(n-r)n!}{(r+1)!(n-r)(n-r-1)!} \\
&= \frac{(r+1)n!}{(r+1)!(n-r)!} + \frac{(n-r)n!}{(r+1)!(n-r)!} \\
&= \frac{(r+1)n! + (n-r)n!}{(r+1)!(n-r)!} \\
&= \frac{n!(r+1+n-r)}{(r+1)!(n-r)!} \\
&= \frac{n!(n+1)}{(r+1)!(n-r)!} \\
&= \frac{(n+1)!}{(r+1)!(n-r)!}
\end{aligned}$$

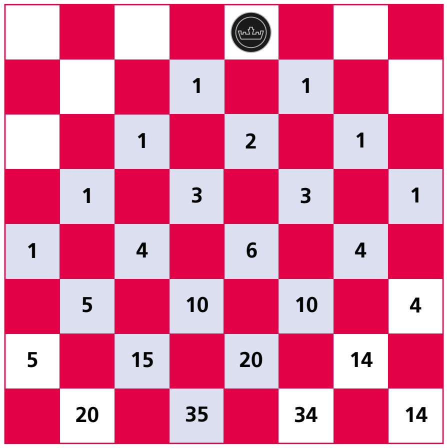
$$\begin{aligned}
\binom{n+1}{r+1} &= \frac{(n+1)!}{(r+1)![(n+1)-(r+1)]!} \\
&= \frac{(n+1)!}{(r+1)!(n-r)!}
\end{aligned}$$

Example 3 Using Pascal's Triangle to Analyze Routes on a Checkerboard

A checker is placed on a game board as shown below. Determine the number of paths the checker may take to get to each allowable square on the board if it can move only diagonally forward one square at a time.

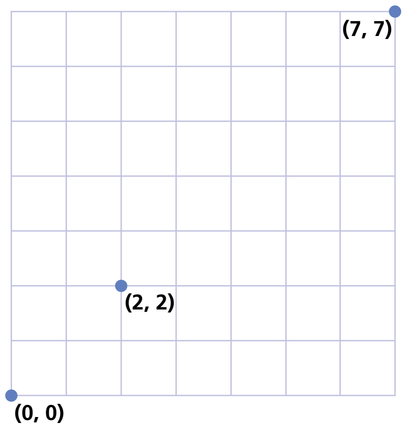


Solution



Indicate in each square the cumulative number of possible routes that led to it. Note the pattern that results. The squares shaded in blue show values found in Pascal's Triangle. The other squares contain values that have been adjusted for terms that were cut off by the edges of the board.

Example 4 Finding the Number of Routes Through a City Street Grid



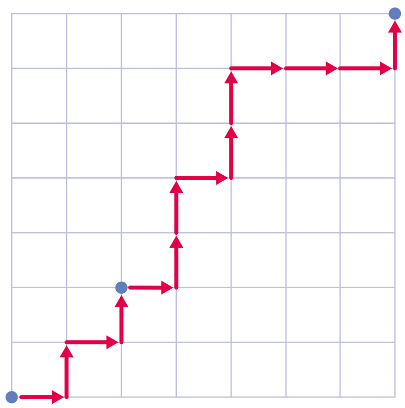
Suppose that you travel without backtracking through the following city street grid, moving only north and east.

Label the bottom left corner as (0, 0) and the top right corner as (7, 7).

(a) How many routes pass through (2, 2)?

(b) How many routes avoid (2, 2)?

Solution



(a) The routes that lead to (2, 2) from (0, 0) must each contain 2 north and 2 east passages. There are $\binom{4}{2}$ of these. An example appears in red in the diagram. The routes from (2, 2) to (7, 7) must contain 5 north and 5 east passages. There are $\binom{10}{5}$ of these. As a result, the total number of routes from (0, 0) to (2, 2) and then on to (7, 7) is the product $\binom{4}{2} \times \binom{10}{5}$, or 1512 routes. One example appears in red in the diagram.

(b) The total number of routes from (0, 0) to (7, 7) without restrictions is $\binom{14}{7}$.

The number of routes that avoid (2, 2) is equal to $\binom{14}{7}$ minus the number of routes that pass through (2, 2) from part (a). Therefore, there are $\binom{14}{7} - \binom{4}{2} \times \binom{10}{5}$, or 1920 routes.

Example 5 Finding a Sum Using the Binomial Theorem

Evaluate $\binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$.

Solution

The sum is the expanded version of $(x + y)^7$ when $x = 1$ and $y = 1$.

$$(1 + 1)^7 = \binom{7}{0}(1)^7 + \binom{7}{1}(1)^6(1) + \binom{7}{2}(1)^5(1)^2 + \binom{7}{3}(1)^4(1)^3 + \binom{7}{4}(1)^3(1)^4 + \binom{7}{5}(1)^2(1)^5 + \binom{7}{6}(1)(1)^6 + \binom{7}{7}(1)^7$$

Therefore,

$$2^7 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$

$$128 = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + \binom{7}{4} + \binom{7}{5} + \binom{7}{6} + \binom{7}{7}$$

The expansion of $(a + b)^n$ can be written in a more compact form $(a + b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$, where $n \in \mathbb{N}$. In this form, individual terms of any binomial expansion can be determined.

General Term of a Binomial Expansion

The general term in the expansion of $(a + b)^n$ is

$$t_{r+1} = \binom{n}{r} a^{n-r} b^r \qquad (r = 0, 1, 2, \ldots, n)$$

This formula can be used to determine the $(r + 1)^{\text{st}}$ term in a binomial expansion.

Example 6 Finding the Term in the Expansion of a Binomial That Is Independent of *x*

Is there a constant term in the expansion of $\left(x + \frac{1}{x}\right)^{10}$ that is independent of x?

Think about The Algebra of Exponents

What properties of exponents are used to simplify the expression to the right?

Solution

The general term in the expansion of this binomial is

$$\begin{aligned} t_{r+1} &= \binom{10}{r} x^{10-r}\left(\frac{1}{x}\right)^r \\ &= \binom{10}{r} x^{10-r}(x^{-1})^r \\ &= \binom{10}{r} x^{10-r}x^{-r} \\ &= \binom{10}{r} x^{10-2r} \end{aligned}$$

For a constant term, the exponent on x must be 0.

Therefore, $$\begin{aligned} 10 - 2r &= 0 \\ r &= 5 \end{aligned}$$

Thus, the constant term in this expansion is $\binom{10}{5}$, or 252.

KEY IDEAS

Binomial Theorem

$$\begin{aligned} (a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots \\ &\quad + \binom{n}{n}b^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r, \text{ where } n \in \mathrm{W} \end{aligned}$$

General Term of a Binomial Expansion—in the expansion of $(a + b)^n$, the general term is of the form $\binom{n}{r}a^{n-r}b^r$, providing the $(r + 1)^{\text{st}}$ term in the expansion of $(a + b)^n$; the coefficients of these terms are often termed **binomial coefficients**

Pascal's Triangle—the coefficients in the expansion of $(a + b)^n$ form a triangular array of numbers known as Pascal's Triangle. These numbers are the binomial coefficients that result from the Binomial Theorem. Problems involving routes through rectangular grid systems and checkerboards or chessboards can be analyzed using the binomial coefficients that appear in Pascal's Triangle.

Pascal's Identity—the coefficients in Pascal's Triangle obey the identity

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

5.2 Exercises

A

1. Use Pascal's Identity to write an expression of the form $\binom{n}{r}$ that is equivalent to each of the following.

(a) $\binom{10}{2} + \binom{10}{3}$ **(b)** $\binom{20}{18} + \binom{20}{19}$

(c) $\binom{15}{14} + \binom{15}{13}$ **(d)** $\binom{n}{r-2} + \binom{n}{r-1}$

2. Write an expression that is equivalent to each of the following.

(a) $\binom{10}{2}$ **(b)** $\binom{20}{18}$ **(c)** $\binom{15}{14}$ **(d)** $\binom{100}{98}$

(e) $\binom{7}{7}$ **(f)** $\binom{20}{0}$ **(g)** $\binom{25}{1}$ **(h)** $\binom{100}{93}$

3. Write the terms in the expansions of the following. Do not simplify your answer.

(a) $(x + y)^6$ **(b)** $(a + b)^5$ **(c)** $(1 - 2)^4$ **(d)** $\left(\frac{2}{3} + \frac{1}{3}\right)^5$

4. **Knowledge and Understanding** In the expansion of $(x + y)^{10}$, write the value of the exponent k in the term that contains

(a) x^4y^k **(b)** x^ky^8 **(c)** x^ky^{4k} **(d)** $x^{k-2}y^{3k}$

5. Express the following in the form $(x + y)^n$.

$$\binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7$$

B

6. If $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 256$, find the value of n.

M

A A

T T T

H H H H

E E E

M M M M

A A A

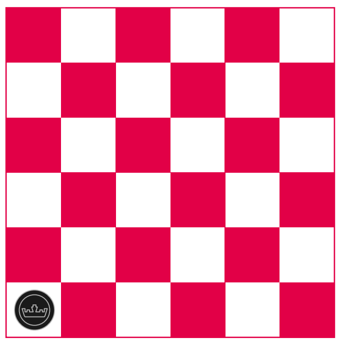
T T

I I I

C C

S

7. Using the arrangement of letters in the margin, compute the number of paths that spell the word MATHEMATICS if all paths must start at the top and move diagonally down through the letters.



8. **(a)** Imagine that a checker is placed in the bottom left corner of a 6-by-6 checkerboard. The piece may be moved one square at a time diagonally left or right to the next row up. Calculate the number of different paths to the top row.

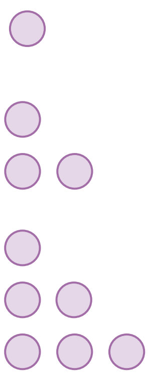
(b) Repeat part (a) with the checker placed in the bottom right corner.

(c) Suppose the checker began in the third square from the left in the bottom row. Calculate the number of possible paths to the top row from this position.

9. Imagine that the terms in each row of Pascal's Triangle had alternating signs.

						1						
					1		–1					
				1		–2		1				
			1		–3		3		–1			
		1		–4		6		–4		1		
	1		–5		10		–10		5		–1	
1		–6		15		–20		15		–6		1

(a) Find the sum of the entries in each row.
(b) Predict the sum for the rows corresponding to $n = 7$, 8, and 9.
(c) Generalize your results to show the value of the sum of
$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}$.



10. **Application** Triangular numbers are based on the figures shown in the margin. Each triangular number corresponds to the number of discs required to construct a triangular pile.
(a) Determine the number of discs in the fourth, fifth, and sixth triangular numbers.
(b) Locate the triangular numbers in Pascal's Triangle.
(c) Find an expression in the form $\binom{n}{r}$ for the nth triangular number.
(d) Use your results to evaluate $1 + 2 + 3 + \ldots + 100$.

11. Using the arrangement of letters to the right, compute the number of paths that spell the word MATH if all paths must start at the top and move diagonally down through the letters.

			M			
		A		A		
	T		T		T	
H		H		H		H

12. **Thinking, Inquiry, Problem Solving** Faizel wants to travel from his house to the hardware store that is six blocks east and five blocks south of his home. If he walks east and south, how many different routes can he follow from his home to the store?

13. Expand and simplify each of the following using the Binomial Theorem.
(a) $(a + 2b)^4$ **(b)** $(x - y)^6$
(c) $\left(c + \frac{1}{c}\right)^4$ **(d)** $\left(d - \frac{1}{d}\right)^5$

14. Find an expression for the general term, in simplified form, for each of the following.
(a) $(x + y)^{10}$ **(b)** $(x - y)^{10}$
(c) $\left(z + \frac{1}{z}\right)^8$ **(d)** $\left(w^2 + \frac{1}{w}\right)^9$

15. Find an expression for the indicated term in the expansion of each of the following.

(a)	$(x^2 - 2)^7$	third term
(b)	$(c - d)^8$	middle term
(c)	$\left(\frac{x}{3} - \frac{3}{x}\right)^{12}$	tenth term
(d)	$\left(y + \frac{1}{y}\right)^{15}$	term independent of y
(e)	$(0.25 + 0.75)^6$	fifth term

C **16.** **Communication** Examine each step in the proof of Pascal's Identity. Explain the algebraic operations used in each step and the purpose of each step.

17. Why is the identity

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

equivalent to the identity

$$\binom{n-1}{r} + \binom{n-1}{r+1} = \binom{n}{r+1}?$$

18. Why are both identities in Question 17 equivalent to

$$\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}?$$

19. In the expansion of $(ax + by)^n$, the coefficients of the first three terms are 6561, 34 992, and 81 648. Find the values of a, b, and n.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

20. **Knowledge and Understanding**

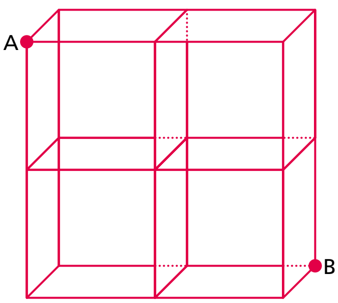
(a) How many terms are there in the expansion of $(3x + 2y)^6$?

(b) Expand $(3x + 2y)^6$.

21. **Application** In the expansion $\left(4x - \frac{2}{x}\right)^8$, find the following.

(a) the term containing x^6

(b) the constant term



22. **Thinking, Inquiry, Problem Solving** Determine the number of possible paths from point A to point B in the diagram to the left if travel may occur only along the edges of the cubes and if the path must always move closer to B.

23. **Communication** Describe the relationship that exists between Pascal's Triangle and the binomial expansion of $(a + b)^n$. Use an example to help illustrate this relationship.