4.7 Counting Techniques and Probability Strategies—Combinations

The previous section showed you that the outcomes of complex experiments can be counted by combining the outcomes of simple experiments when order matters. For many counting problems, order is not important. For example, in most card games, the order in which the cards are dealt is not important. Rearranging the cards that you have been dealt does not change your hand. This section will examine counting techniques and probability problems that involve combinations.

SELECTING OBJECTS WHEN ORDER DOES NOT MATTER

Example 1 Calculating Combinations

Suppose you have nine children who want to play a game that requires three players at a time. In how many ways can you choose a team of three children? The order in which you select the children does not matter.

Think about Order
Why doesn't the order in which they are selected matter?

Solution

Suppose that teams are selected as follows:

	First Choice	Second Choice	Third Choice
Team A	Ben	Mary	Amir
Team B	Mary	Amir	Ben
Team C	Amir	Ben	Mary

The total number of three permutations of the nine children is P(9, 3) $= 9 \times 8 \times 7$ or 504. This is the number of all possible three-person teams in which order matters.

combination—an unordered selection of elements from a given set

Each of the teams shown above is essentially the same. In total, Ben, Mary, and Amir could be arranged in P(3, 3) $=$ 3! or 6 ways. Therefore, the number of distinct teams, ignoring the order in which players are chosen, is $\frac{P(9, 3)}{3!} = \frac{504}{6}$ or 84. When order does not matter, you divide the total number of permutations by the number of like arrangements. This determines the number of unique combinations.

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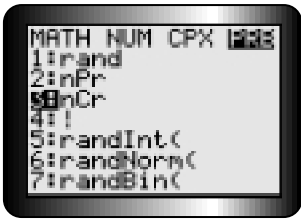
Combinations can be computed on a TI-83 Plus calculator using the nCr command. See Appendix C.14 on page 410.



Think about C(n, r)
People often read C(n, r) as "n choose r." Why does this seem reasonable?

Combination

A combination is a collection of chosen objects for which order does not matter. C(n, r)—sometimes written as ${}_nC_r$ or as $\binom{n}{r}$—represents the number of combinations possible in which r objects are selected from a set of n different objects.

$$\begin{aligned} C(n, r) &= \frac{P(n, r)}{P(r, r)} \\ &= \frac{P(n, r)}{r!} \\ &= \frac{n!}{(n-r)!\,r!} \end{aligned}$$

For example,

$$\begin{aligned} C(9, 3) &= {}_9C_3 \\ &= \binom{9}{3} \\ &= \frac{9!}{6!\,3!} \\ &= \frac{9 \times 8 \times \ldots 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} \\ &= 84 \end{aligned}$$

Example 2 Calculating the Number of Combinations

From a class of 30 students, determine how many ways a five-person committee can be selected to organize a class party

(a) with no restrictions

(b) with Marnie on the committee

Solution

(a) Each person is not assigned a specific responsibility, so the order does not matter.

$$\begin{aligned} \text{number of committees} &= C(30, 5) \\ &= \binom{30}{5} \\ &= \frac{30!}{25!\,5!} \\ &= 142\ 506 \end{aligned}$$

(b) With Marnie on the committee, there are $\binom{29}{4} = 23\ 751$ ways of choosing the remaining members.

Think about Restrictions
What restriction must be satisfied before the committee members are selected in part (b)?

Example 3 Calculating the Number of Combinations

Tanya Kozovski, the coach of a co-ed basketball team, must select five players to start the game from a team that consists of six females and five males. How many ways can this be achieved if Tanya must choose three females and two males to start the game?

Think about The Selections

Why are the number of ways that the coach can pick males and females multiplied?

Solution

Tanya can choose the three females in $\binom{6}{3}$ ways and the males in $\binom{5}{2}$ ways. Thus, the number of ways to choose the starting line-up is as follows:

$$\binom{6}{3} \times \binom{5}{2} = 20 \times 10$$
$$= 200 \text{ ways}$$

Example 4 Calculating the Number of Combinations

In how many ways can 6 people be selected from a group that consists of four adults and eight children if the group must contain at least two adults?

direct reasoning—all suitable outcomes are totalled to arrive at a final answer

Solution 1 *Direct Reasoning*

In this situation, the condition that the group must have at least two adults must be satisfied. This can happen three ways: a group with two adults and four children; three adults and three children; or four adults and two children.

$$\text{number of ways} = \binom{4}{2}\binom{8}{4} + \binom{4}{3}\binom{8}{3} + \binom{4}{4}\binom{8}{2}$$
$$= 6 \times 70 + 4 \times 56 + 1 \times 28$$
$$= 420 + 224 + 28$$
$$= 672$$

indirect reasoning—undesired outcomes are subtracted from the total to arrive at a final answer

Solution 2 *Indirect Reasoning*

In this situation, the solution can also be found by subtracting the number of ways the condition is not satisfied (0 or 1 adult in the group) from the total number of combinations.

$$\text{number of ways} = \binom{12}{6} - \binom{4}{0}\binom{8}{6} - \binom{4}{1}\binom{8}{5}$$
$$= 924 - 1 \times 28 - 4 \times 56$$
$$= 924 - 28 - 224$$
$$= 672$$

Example 5 Using Combinations to Find Probabilities

Five cards are dealt at random from a deck of 52 playing cards. Determine the probability that you will have

(a) the 10–J–Q–K–A of the same suit

(b) four of a kind

Solution

(a) The number of five-card hands possible from a deck of 52 cards is $\binom{52}{5} = 2\ 598\ 960$. There are only four suits, so there are only four hands that are 10–J–Q–K–A of the same suit.

$P(\text{10–J–Q–K–A same suit}) = \frac{4}{2\ 598\ 960}$ or $\frac{1}{649\ 740}$

Think about $\binom{4}{4}$

- Why does $\binom{4}{4} = 1$?
- Why does $\binom{4}{0} = 1$?
- Why are these answers reasonble?

(b) There are 13 cards in each suit and the hand must contain four of the same card. The remaining card in the hand can be any card of the remaining 48 in the deck.

$$\begin{aligned} P(\text{four of a kind}) &= \frac{13\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} \\ &= \frac{13 \times 1 \times 48}{2\ 598\ 960} \\ &= \frac{624}{2\ 598\ 960} \\ &= \frac{1}{4165} \end{aligned}$$

PROBABILITY AND ODDS

The terms *probability* and *odds* are often used interchangeably. However, they mean two different things.

Suppose there are four white balls and seven black balls in a bag. You need to reach into the bag and select one ball.

- The probability that you will select a white ball is $\frac{4}{11}$.
- The odds that you will select a white ball is 4 to 7 or 4:7.

Calculating odds involves comparing the number of favourable outcomes to the number of unfavourable outcomes. This is different from probability, which involves comparing the number of favourable outcomes to the total number of possible outcomes.

KEY IDEAS

combination—a collection of chosen objects for which order does not matter

C(*n*, *r*)—sometimes written as $_nC_r$ or as $\binom{n}{r}$—represents the number of combinations possible in which *r* objects are selected from a set of *n* different objects

$$\begin{aligned} C(n, r) &= \frac{P(n, r)}{P(r, r)} \\ &= \frac{P(n, r)}{r!} \\ &= \frac{n!}{(n-r)!\,r!} \end{aligned}$$

When order doesn't matter in a complex probability question, use the appropriate formulas for combinations to determine the number of ways the event can occur and the total number of possible outcomes.

direct reasoning—all suitable outcomes are totalled to arrive at a final answer

indirect reasoning—undesired outcomes are subtracted from the total to arrive at a final answer

odds—number of favourable outcomes : number of unfavourable outcomes

4.7 Exercises

A

1. Evaluate each of the following.

(a) C(8, 3) **(b)** ${}_7C_4$ **(c)** $\binom{12}{11}$

(d) $\binom{5}{2}$ **(e)** C(10, 3) **(f)** $\binom{15}{15}$

2. In how many ways can a team of six female volleyball players be chosen to start the game from a roster of 12 players?

3. In how many ways can a principal select a graduation committee consisting of two teachers and four students if there are six teachers and ten students who are volunteering for the positions?

4. **Knowledge and Understanding** A bag contains 10 red jellybeans and 8 black jellybeans.

(a) Determine the number of ways that 2 jellybeans can be chosen from the 18 that are in the bag.
(b) Determine the number of ways that 2 red jellybeans can be selected.
(c) What is the probability that 2 jellybeans selected at random are red?

B

5. In the card game Crazy Eights, how many different eight-card hands can be dealt from a standard 52-card deck?

6. Show that

(a) C(10, 5) = C(9, 4) + C(9, 5) **(b)** $5\binom{n}{5} = n\binom{n-1}{4}$

(c) $r\binom{n}{r} = n\binom{n-1}{r-1}$ **(d)** $2\binom{2n-1}{n-1} = \binom{2n}{n}$

7. From a group of five men and four women, determine how many committees of five people can be formed with

(a) no restrictions **(b)** exactly three women
(c) exactly four men **(d)** no women
(e) at least two men **(f)** at least three women
(g) Find the probability of each of parts (b) to (f).

8. **(a)** A rooming house has three rooms that contain four beds, three beds, and two beds, respectively. In how many ways can nine guests be assigned to these rooms?
(b) What is the probability that Renaldo will be assigned to the room with two beds?

9. Three cards are selected at random from a standard deck of 52 playing cards. Determine the probability that all three cards are

(a) hearts **(b)** black **(c)** aces **(d)** face cards

10. A paper bag contains a mixture of three types of candy. There are ten gum balls, seven candy bars, and three packages of toffee. Suppose a game is played in which a candy is randomly taken from the bag and then a second candy is drawn from the bag, without replacement. You are allowed to keep both candies if, and only if, the second is the same type as the first.

(a) Calculate the probability that you will be able to keep a gum ball on the first try.
(b) Calculate the probability that you will be able to keep any candy on the first try.
(c) Calculate the probability that you will not be able to keep any candy on the first try.

11. **Application** The odds in favour of an event are expressed as the ratio

A′—the complement of *A*

$$P(A):P(A') = P(A):(1 - P(A))$$

The winning numbers for the Lotto 6/49 lottery are drawn from a clear plastic drum that contains 49 ping-pong balls numbered from 1 to 49. The order of selection does not matter. Once a ball is drawn from the drum, it is put on display. The process is repeated a total of six times. You can play the lottery by having the computer randomly pick a combination of six numbers for you. What are the odds in favour of you winning the jackpot (matching all six numbers) in this way?

12. A single coin is tossed.

(a) What is the probability of tossing a head? A tail?
(b) What are the odds of tossing a head? A tail?
(c) Describe how the probability of an event is similar to the odds of an event occuring.

13. Melik has five quarters and six dimes in his pocket. He pulls out one coin.

(a) What are the odds of the coin being a quarter?
(b) What are the odds of the coin being a dime?

14. Suppose the probability of rain tomorrow is 80%. What are the odds of rain tomorrow?

15. The coach says that the probability of winning the next game is 40%. What are the odds the team will win?

16. People often talk about the odds *for* or *against* an event. For example, you saw on page 261 that the odds of selecting a white ball were 4:7. The odds against are, thus, 7:4.

(a) In a horse race, odds are not written in the standard way. For example, a horse with odds 100:1 has little chance of winning. Explain how the odds work.

(b) Suppose the odds are 100:1 that a horse will win a race. What is the probability the horse will win?

17. **Communication**

(a) How are combinations and permutations similar? How are they different? Use examples in your answer.

(b) Explain what ${}_nC_r$ and ${}_nP_r$ represent. How can you find ${}_nC_r$ if you know ${}_nP_r$? Use an example in your answer.

C **18.** A CD player can hold five different CDs. The chart below shows the number of songs on each CD in the player.

CD	1	2	3	4	5
Number of Songs	13	15	11	12	16

If the player is set on shuffle and randomly selects songs to play from the five discs, calculate the probability that during the first five songs played

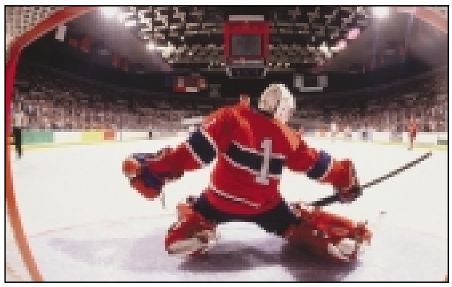
(a) they will be from CDs 1 and 5

(b) one song from each of the CDs is played

(c) your favourite song from each of the five CDs is played

19. **Thinking, Inquiry, Problem Solving**

In a female minor hockey league of 10 teams from different cities, each pair of teams must play three games. Can a schedule be created so that the same number of games is played in each of the 10 cities? Justify your answer.



20. Algebraically prove that the following is true for $n \geq 3$. Why must the condition on the value of n be included?

$$\binom{n}{3} + \binom{n}{2} + \frac{1}{6}\binom{n}{1} = \frac{n^3}{6}$$

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

21. **Knowledge and Understanding** A photographer for an advertising photo shoot has a group of models available consisting of three male adults, four female adults, and five children. In how many ways can the photographer choose four models if there must be one adult male, one adult female, and two children?

22. **Application** Determine the probability that a four-card hand dealt from a standard deck of 52 playing cards contains a card from each suit.

23. **Thinking, Inquiry, Problem Solving** Prove that $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$.

24. **Communication**

(a) Suppose a commitee of 3 is to be selected from a group of 10 people.

(b) Suppose the committee is actually an executive committee consisting of a president, vice-president, and secretary.

Explain why counting the number of committees described in part (a) will use combinations while those in part (b) will require the use of permutations.

Chapter Problem

Analyzing a Traditional Game

CP16. Use the combination formula to verify that there are 2^6 possible combinations possible when the counters are tossed.

CP17. Copy and complete the table by calculating the probability of each of the following occurring on any random turn and the points to be awarded.

Outcome	Probability	Points
6 Blue, 0 Red		
5 Blue, 1 Red		
4 Blue, 2 Red		
3 Blue, 3 Red		
2 Blue, 4 Red		
1 Blue, 5 Red		
0 Blue, 6 Red		

CP18. Multiply the probability by the number of points, add up the results, and simplify. This is the expected or average number of points you should earn on each turn.

CP19. Based on your findings, how many turns would you expect a typical game to have before someone wins?