

4.6 Counting Techniques and Probability Strategies—Permutations

The previous sections showed you that the outcomes of complex experiments can be counted by combining the outcomes of simple experiments. This section will present the mathematical tools and strategies that will allow you to count these outcomes efficiently, and then find the probabilities for these events when the order of the simple events is important.

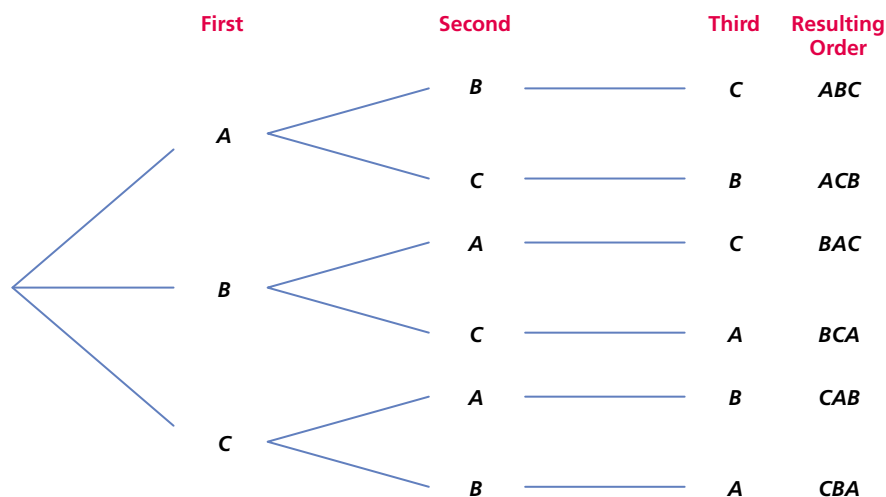
Example 1 Creating Game Line-Ups

You are trying to put three children—represented by A , B , and C —in a line for a game.

- How many different orders are possible?
- What is the probability that a random ordering will produce the order ABC ?

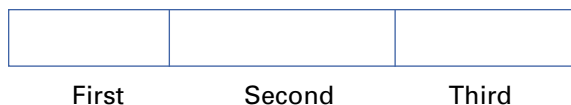
Solution

- Construct a tree diagram for this sequence of choices using A , B , and C to represent the children.



There are six possible arrangements.

Another way of visualizing this problem is to think of placing each child's name into one of three different boxes on a piece of paper.



? Think about The Choices

Why is there only one available choice for third after first and second have been determined?

? Think about Order Decisions

Why do the tree diagram and the box diagram represent the same set of choices that can be made?

There are three choices for filling in the first box. Once a name has been placed in that box, there are two ways of filling in the second box. Once that has been done, there is one remaining way to fill in the third box. The Multiplicative Principle states that there are $3 \times 2 \times 1$, or six ways of doing this.

- (b) Of the six arrangements that are possible, only one of them is the arrangement ABC .

$$P(ABC) = \frac{1}{6}$$

Discussion Questions

- Suppose you have nine players who want to play on the team, but you are allowed to select only three for any game.
 - How many different three-player line-up orders can you create using the nine available players?
 - How many different six-player line-up orders can you create using the nine available players?
- Suppose that you are the manager of a female baseball team with 15 players. During a game, nine players are used. How many different batting orders can you create for the nine players who start the game?
- Children frequently play with interlocking plastic blocks. These blocks come in red, blue, white, green, grey, and black. Suppose you have one block of each colour. The position of the locking pegs and holes allows you to determine which block starts a chain.
 - How many different six-block sequences can be formed?
 - How many different two-block sequences can be formed?
 - Why is it important to determine which block starts a chain?



Think about Question 2

Explain why a tree diagram would not be a very efficient way to solve Question 2.

SELECTING OBJECTS WHEN ORDER MATTERS

Factorial Notation

The ordering problem in Example 1 dealt with arranging the names of three children to create sequences with different orders. In the simplest case, we used all the children's names to fill the same number of positions. This was done by making a first choice in which all names could be chosen. Then, the second choice had one fewer choice available, since one child's name had already been selected and was no longer available. Finally, the last choice was forced since there was only one remaining name. Thus, there are $3 \times 2 \times 1$ ways of putting the children in a line. This can be written as $3!$.

factorial notation— $n!$ represents the number of ways that n different objects can be selected to create ordered arrangements of size n



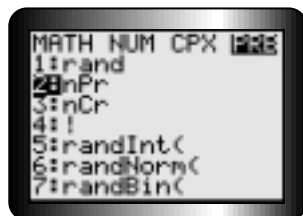
Think about The Notation

- Why does each term in the expansion of $n!$ decrease by 1 until the last term is 1?
- $0!$ is defined to have a value of 1. Why does this definition make sense?



Technolink

Permutations can be computed on a TI-83 Plus calculator using the nPr command. See Appendix C.14 on page 410.



Think about Permutations

Why is the following statement true?

$$P(n, n) = n!$$

Factorial Notation

$n!$ (read as n factorial) is calculated as

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

For example,

$$\begin{aligned} 5! &= 5 \times 4 \times 3 \times 2 \times 1 \\ &= 120 \end{aligned}$$

Permutations

Suppose you have n objects to choose from, but only want to select some rather than all of them. There is still a sequence of choices to be made. In this case, not all the elements in the set are selected.

Permutation

A *permutation* is an ordered arrangement of objects selected from a set. $P(n, r)$ (sometimes written as ${}_nP_r$) represents the number of permutations possible in which r objects from a set of n different objects are arranged.

$$\begin{aligned} P(n, r) &= n(n - 1)(n - 2) \dots (n - r + 1) \\ &= [n(n - 1)(n - 2) \dots (n - r + 1)] \times \left[\frac{(n - r)(n - r - 1) \dots 3 \times 2 \times 1}{(n - r)(n - r - 1) \dots 3 \times 2 \times 1} \right] \\ &= \frac{n(n - 1)(n - 2)(n - 3) \dots (n - r + 1) \times (n - r)(n - r - 1) \dots 3 \times 2 \times 1}{(n - r)(n - r - 1) \dots 3 \times 2 \times 1} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

For example,

$$\begin{aligned} P(5, 3) &= \frac{5!}{(5 - 3)!} \\ &= \frac{120}{2} \\ &= 60 \end{aligned}$$

That means there are 60 ways of ordering objects taken three at a time from a set of five different objects.

Example 2 Calculating Permutations

There are 15 players on the school baseball team. How many ways can the coach complete the nine-person batting order?

Solution

Whenever you count the ways that different objects can be put into arrangements of any size, the order in which the objects are selected is an essential considera-

tion. Order matters in this situation. We must calculate the number of arrangements that contain 9 items from a set that contains 15.

$$\begin{aligned}
 P(15, 9) &= \frac{15!}{(15 - 9)!} \\
 &= \frac{15!}{6!} \\
 &= \frac{15 \times 14 \times 13 \times \dots \times 2 \times 1}{6 \times 5 \times \dots \times 2 \times 1} \\
 &= 15 \times 14 \times 13 \times \dots \times 7 \\
 &= 1\,816\,214\,400
 \end{aligned}$$

CREATING PERMUTATIONS WHEN SOME OF THE OBJECTS ARE ALIKE

How many distinctly different “words” can be formed using the letters of the word OTTAWA?

Suppose it was possible to mark each of the repeated letters so that the two T’s and the two A’s could be distinguished from one another. By colouring the repeated letters in OTTAWA, we create six letters that are different from one another. These can be arranged in $6!$ different ways.

One such arrangement is AOTWAT and another is AOTWAT. If the colouring is removed, these two arrangements form the same “word.” Since there are $2!$ ways to arrange the T’s, and for each of these there are $2!$ ways to arrange the A’s without actually changing the word, there will be $2! \times 2!$ duplicates of the same word. To correct this overcounting of identical arrangements, divide $6!$ by the number of ways the duplicate words can be formed for each possible ordering of the letters. Therefore, the number of distinctly different “words” is given by $\frac{6!}{2!2!}$.

Number of Permutations

$\frac{n!}{a!b!c!\dots}$ represents the number of permutations from a set of n objects in which a are alike, b are alike, c are alike, and so on.

Example 3 Calculating Permutations When Objects Are Alike

Determine the number of arrangements possible using the letters of the word MATHEMATICS.

Solution

There are 11 letters and there are two M’s, two A’s, and two T’s. Therefore, the number of arrangements is $\frac{11!}{2!2!2!} = 4\,989\,600$.

SELECTING OBJECTS WITH AND WITHOUT REPLACEMENT

In the situations examined so far, objects were selected from a set and then, once selected, were removed from the collection so that they could not be chosen again. If the object is replaced, how does this affect the possible number of arrangements?

The Problem

How many ways are there to draw two cards from a standard deck of 52 cards?

Analysis and Solution

If you draw the cards without replacement, you draw one card and then, without putting the first card back in the deck, you draw another. You then note what the two cards are. There are 52 ways of selecting the first card, and 51 ways of selecting the second card once the first has been chosen. Therefore, there are $52 \times 51 = P(52, 2)$ or 2652 ways of selecting two cards this way.

If you draw the cards with replacement, you draw a card, note it, and then put it back in the deck. You then shuffle the deck and draw another card. There are 52 ways of selecting the first card. For each possible first card, there are 52 ways of selecting the second card. There are $52 \times 52 = 52^2$ or 2704 ways of selecting the two cards this way.

Replacement increases the number of possible choices.

Example 4 Using Permutations to Determine Probability

Four people are required to help out at a party: one to prepare the food, one to serve it, one to clear the tables, and one to wash up. Determine the probability that you and your three siblings will be chosen for these jobs if four people are randomly selected from a room of 12 people.

Solution

$$\begin{aligned} P(\text{you and siblings selected}) &= \frac{n(\text{you and your siblings can be chosen for the four jobs})}{n(12 \text{ people can be chosen for the four jobs})} \\ &= \frac{4!}{{}_{12}P_4} \\ &= \frac{4!}{\frac{12!}{(12-4)!}} \\ &= \frac{4!}{\frac{12!}{8!}} \\ &= \frac{4 \times 3 \times 2 \times 1}{12 \times 11 \times 10 \times 9} \\ &= \frac{1}{495} \end{aligned}$$

KEY IDEAS

factorial notation ($n!$)

- represents the number of ordered arrangement of n objects

$$n! = n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$$

- represents the number of permutations possible from a set of n different objects
- $0! = 1$

permutations

- $P(n, r)$ (sometimes written as ${}_nP_r$) represents the number of permutations possible in which r objects from a set of n different objects are arranged.
- $P(n, r) = \frac{n!}{(n - r)!}$
- $\frac{n!}{a!b!c!\dots}$ represents the number of permutations from a set of n objects in which a are alike, b are alike, c are alike, and so on.
- When order matters in a probability question, use the appropriate formulas for permutations to determine the number of ways the event can occur and the total number of possible outcomes.

4.6 Exercises

A

1. Evaluate each of the following.

(a) $5!$

(b) ${}_5P_3$

(c) $8! \times 6!$

(d) $\frac{5!}{4!}$

(e) $P(10, 3)$

(f) $\frac{16!}{5!6!8!}$

2. Simplify each of the following.

(a) $\frac{n!}{(n - 1)!}$

(b) $\frac{(3n)!}{(3n - 1)!}$

(c) $\frac{(n - r)!}{(n - r - 1)!}$

3. Express the following using factorials.

(a) $5 \times 4 \times 3 \times 2 \times 1$

(b) $8 \times 7 \times 6$

(c) $\frac{30 \times 29 \times 28}{3 \times 2 \times 1}$

(d) 12×11

4. (a) In how many ways can 10 students standing in a line be arranged?
(b) In how many ways can 10 students standing in a line be arranged if Jill must be first?
(c) In how many ways can 10 students standing in a line be arranged if Jill must be first and Meera last?

5. In how many ways is it possible to elect a president, a vice-president, and a secretary for a club consisting of 15 members?
6. **Knowledge and Understanding** In how many ways can the letters of the word MONDAY be arranged if
 - (a) all six letters are used
 - (b) only four of the letters are used
7. In how many different ways can the letters of the word MISSISSAUGA be arranged?

B

8. (a) Show that ${}_6P_4 = 6({}_5P_3)$.
 (b) Solve the equation for n .

(i) $\frac{n!}{(n-1)!} = 5$	(ii) $\frac{n!}{(n-2)!} = 90$
(iii) ${}_nP_5 = 14({}_nP_4)$	(iv) ${}_nP_3 = 17({}_nP_2)$
9. A standard deck of cards has had all the face cards (jacks, queens, and kings) removed so that only the ace through ten of each suit remains. A game is played in which two cards are drawn (without replacement) from this deck and a six-sided die is rolled. For the purpose of this game, an ace is considered to have a value of 1.
 - (a) Determine the total number of possible outcomes for this game.
 - (b) Find the probability of each of the following events.
 - (i) one even card is drawn and an even number is rolled
 - (ii) two even cards are drawn and an odd number is rolled
 - (iii) one card of 3 is drawn and 3 or less is rolled
 - (iv) the sum of the cards and the die is 7
 - (v) the sum of the cards and the die is less than 5
10. Repeat Question 9 if the game is played with replacement after the first card is drawn.
11. **Application** A combination lock opens when the right combination of three numbers from 00 to 99 is entered. The same number may be used more than once.
 - (a) What is the probability of getting the correct combination by chance?
 - (b) What is the probability of getting the right combination if you already know the first digit?
12. A bag contains four red, three green, and five yellow marbles. Three marbles are drawn, one at time, without replacement. Determine the probability that the order in which they are selected is
 - (a) yellow, red, green
 - (b) yellow, green, green
 - (c) yellow, yellow, red
13. **Communication** You are taking a chemistry test and are asked to list the first 10 elements of the periodic table in order as they appear in the table. You know the first 10 elements but not the order. Explain why the probability of guessing the correct answer is $\frac{1}{3\,628\,800}$.



- C** 14. **Thinking, Inquiry, Problem Solving** Several years ago, regular motor-vehicle licence plates comprised three letters followed by three numbers. Later, any combination of six characters was permitted in order to provide a greater supply of available licence plates.
- (a) By how much did the supply of possible licence-plate numbers increase?
 - (b) Plates issued by the Motor Vehicle License Office now use four letters followed by three numbers. How many such plates are there now?
 - (c) For an extra charge, it is possible for car owners to purchase vanity plates that can have up to seven characters on them. How many seven-character plates are possible, excluding those that fit the pattern of the plates described in part (b)? (**Note:** The Ministry of Transportation places some restrictions on the use of characters and words on the licence plates. As a result, the number of allowable licence plates is less than the number of possible plates.)
15. Solve for n : $\frac{(n-1)!}{(n-3)!} = 20$.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

16. **Knowledge and Understanding** Twelve students have signed up to serve on the yearbook committee.
- (a) How many ways can the staff adviser choose three people to act as publisher, lead photographer, and editor, respectively?
 - (b) In filling these three positions, what is the probability that Francesca is selected as publisher and Marc is chosen as editor?
17. **Application** The moderator of a forum discussion is assigning seats to the six participants. If the seats are arranged in a linear fashion, determine the probability that Dr. Eisen and Dr. Bugada are seated next to each other.
18. **Thinking, Inquiry, Problem Solving** How many three-letter arrangements are possible using the letters of the word CANADA?
19. **Communication** Create a permutation question that also involves probability. Determine the solution to your problem.

Chapter Problem

Analyzing a Traditional Game

- CP14.** In this game, all six counters are tossed simultaneously. Does order play a role in determining the number of outcomes? Explain.
- CP15.** If the rules of the game were changed so that the counters were tossed one at a time, would this affect the probabilities associated with earning points? Explain.