4.5 Finding Probability Using Tree Diagrams and Outcome Tables

Games of chance often involve combinations of random events. These might involve drawing one or more cards from a deck, rolling two dice, or tossing coins. How do you find the probability that a particular sequence of these events will occur?

This section will show you some strategies for listing the outcomes of an experiment in an organized way so that you can find the probability of a particular compound event.

TREE DIAGRAMS AND THE MULTIPLICATIVE PRINCIPLE FOR COUNTING OUTCOMES

A tree diagram is often used to show how simple experiments can be combined one after the other to form more complex experiments. The successive branches of the tree each correspond to a step required to generate the possible outcomes of an experiment. The tree diagram and outcome table that correspond to tossing two coins appear below.

Tree Diagram

First Coin	Second Coin	Outcome
Heads	Heads	HH
	Tails	HT
Tails	Heads	TH
	Tails	TT
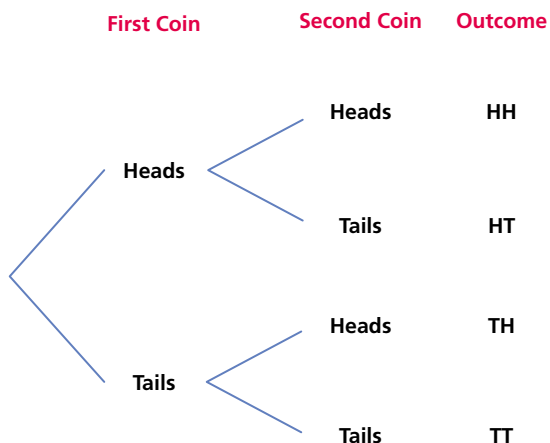
Outcome Table

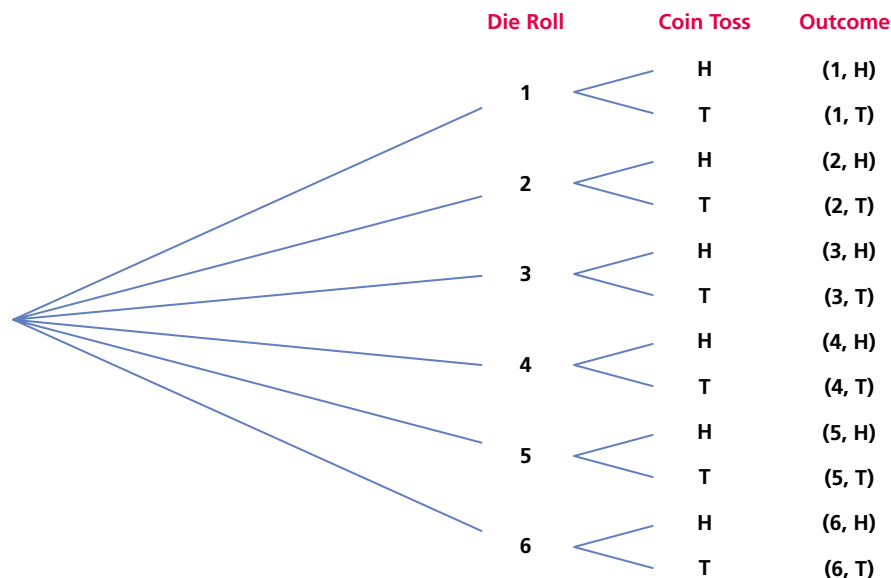| Coin 1	Coin 2	Simple Event
H	H	HH
H	T	HT
T	H	TH
T	T	TT

? Think about
Tree Diagrams
How would the tree diagram differ if the experiment consisted of rolling a die twice instead of tossing a coin twice?

Consider an experiment in which you first roll a six-sided die and then toss a coin. What is the probability of tossing tails and rolling an even number?

The following tree diagram can be used to represent the possible results.

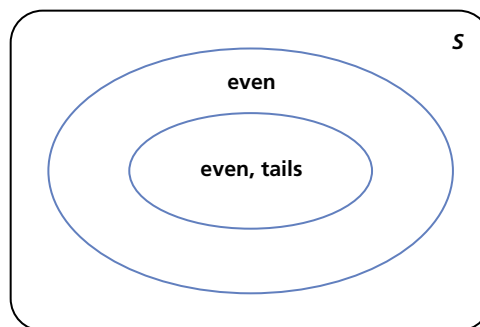| Die Roll	Coin Toss	Outcome
1	H	(1, H)
	T	(1, T)
2	H	(2, H)
	T	(2, T)
3	H	(3, H)
	T	(3, T)
4	H	(4, H)
	T	(4, T)
5	H	(5, H)
	T	(5, T)
6	H	(6, H)
	T	(6, T)

The sample space for this experiment consists of ordered pairs of the form (d, c), where d is the result of a roll of the die and c is the result of the toss of a coin. The tree diagram for the experiment clearly shows the 12 possible outcomes.

For each roll of the die (the first entry in the ordered pair), we have a choice of six possible outcomes. For each of these, there are two choices for the coin toss (the second entry of the ordered pair). As a result, there are $6 \times 2 = 12$ possible outcomes.

There are three outcomes that correspond to the event of an even roll of the die followed by a toss of tails. They are (2, T), (4, T), and (6, T). Therefore, $P(\text{even roll, tails}) = \frac{3}{12}$, or $\frac{1}{4}$.

In this case, the event of getting an even roll of the die and the event of getting tails on a coin toss have no influence on each another.

The following Venn diagram shows the relationship between the events.

INDEPENDENT AND DEPENDENT EVENTS

In the die roll/coin toss experiment,

$$P(\text{even roll, tails}) = \frac{n(\text{even roll, tails})}{n(\text{ordered pairs})}$$

$$= \frac{n(\text{even roll}) \times n(\text{tails})}{n(\text{die rolls}) \times n(\text{coin tosses})}$$

$$= \frac{n(\text{even roll})}{n(\text{die rolls})} \times \frac{n(\text{tails})}{n(\text{coin tosses})}$$

$$= P(\text{even roll}) \times P(\text{tails})$$

$$= \frac{3}{6} \times \frac{1}{2}$$

$$= \frac{3}{12}$$

$$= \frac{1}{4}$$

What is the probability of tossing the coin and getting tails if you know in
advance that the die will show an even number?

    The rolling of the die and the tossing of the coin are independent events.
Knowing the outcome of the die roll has no effect on the probability of
getting tails.

    Consider the conditional probability $P(\text{tails} \mid \text{even})$. From the previous
section, we know that

$$P(\text{tails} \mid \text{even}) = \frac{P(\text{tails} \cap \text{even})}{P(\text{even})}$$

$$= \frac{\left(\frac{3}{12}\right)}{\left(\frac{3}{6}\right)}$$

$$= \frac{1}{2}$$

We also know that $P(\text{tails}) = \frac{1}{2}$. Therefore, $P(\text{tails} \mid \text{even}) = P(\text{tails})$.

**Project
Connection**
The purpose of
your project may
be to determine
whether or not
some variable has
a significant
effect on another
variable, or using
the language of
probability, you
may want to
know whether the
variables are
dependent or not.

Conditional Probability for Independent Events

If *A* and *B* are **independent events**, then

$$P(B \mid A) = P(B)$$

If this is not the case, the events are **dependent**.

Recall that the conditional probability of event *B* given event *A* has occurred is

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

If events *A* and *B* are independent, then $P(B \mid A) = P(B)$. Therefore,
$P(B) = \frac{P(B \cap A)}{P(A)}$ and $P(B \cap A) = P(B) \times P(A)$.

Multiplicative Principle for Probabilities of Independent Events

If *A* and *B* are independent events, then

$$P(A \text{ and } B) = P(A \cap B)$$
$$= P(A) \times P(B)$$

Example 1 The Even–Odd Game

The Even−Odd game is a game that can be played by two players using two
coins. One of the players assumes the Even role; the other, Odd. Each player
tosses a coin. If the two coins show an even number of heads, Even wins. If the
coins show an odd number of heads, Odd wins. A tie is declared if no heads
show.

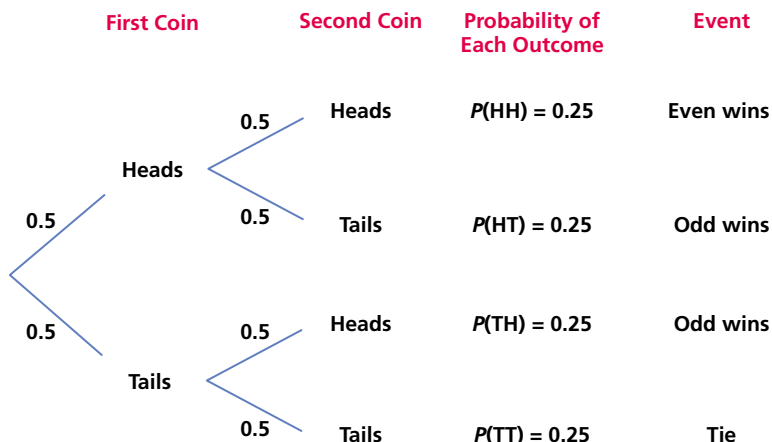Draw a tree diagram to represent the possible events.

Solution

The result of the game—Even wins, Odd wins, or they tie—is a random variable
since it depends on the outcomes of the coin tosses.

Consider the following tree diagram showing the probabilities of the simple
and compound events.

First Coin	Second Coin	Probability of Each Outcome	Event
	0.5 → Heads	$P(HH) = 0.25$	Even wins
Heads			
0.5	0.5 → Tails	$P(HT) = 0.25$	Odd wins
	0.5 → Heads	$P(TH) = 0.25$	Odd wins
0.5 **Tails**			
	0.5 → Tails	$P(TT) = 0.25$	Tie

Example 2 Rolling Two Dice

Suppose you are playing a game in which you roll a red and a black six-sided die at the same time. Consider the event that the sum of the dice is 3. Is this event independent of the red die showing 1?

Solution

A sum of 3, with the red die showing 1, can only happen in one way, if the black die shows 2. There are 36 possible outcomes with the two dice. Therefore,

$$P(\text{red die shows 1 and sum is 3}) = \frac{1}{36}$$

However,

$$P(\text{red die shows 1}) = \frac{1}{6} \text{ and } P(\text{sum is 3}) = \frac{2}{36} \text{ or } \frac{1}{18}$$

Therefore,

$$P(\text{red die shows 1}) \times P(\text{sum is 3}) = \frac{1}{6} \times \frac{1}{18} \text{ or } \frac{1}{108}$$

As a result,

$$P(\text{red die shows 1 and sum is 3}) \neq P(\text{red die shows 1}) \times P(\text{sum is 3})$$
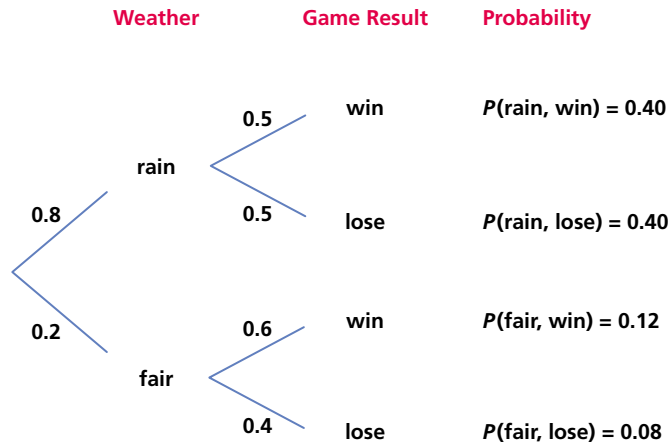
and the events must be dependent.

Example 3 Predicting Soccer Results

Suppose that a school's female soccer team has a history of winning 50% of the games it plays on rainy days and 60% of the games it plays on fair days. The weather forecast for the next game day calls for an 80% chance of rain. What is the probability that the team will win the game?

Solution

Draw a tree diagram that represents the sequence of outcomes as ordered pairs in the form (weather, game result).

Weather	Game Result	Probability

                                          0.5   win   P(rain, win) = 0.40
                              rain
                    0.8   0.5   lose   P(rain, lose) = 0.40

                    0.2   0.6   win   P(fair, win) = 0.12
                              fair
                                          0.4   lose   P(fair, lose) = 0.08

There are two simple events that result in a win. The probabilities for these events are as follows:

$$P(\text{rain, win}) = P(\text{rain}) \times P(\text{win} \mid \text{rain})$$
$$= 0.8 \times 0.5$$
$$= 0.4$$

$$P(\text{fair, win}) = P(\text{fair}) \times P(\text{win} \mid \text{fair})$$
$$= 0.2 \times 0.6$$
$$= 0.12$$

The probability the team will win is, therefore, the sum $0.4 + 0.12 = 0.52$.

? Think about
The Probabilities
Why is the probability of the pair (rain, win) equal to the product 0.8×0.5?

KEY IDEAS

independent events—two events are independent of each other if the occurrence of one event does not change the probability of the occurrence of the other event

conditional probability for independent events—if A and B are independent events, then $P(A \mid B) = P(A)$; if this is not the case, the events are **dependent** and the formula for conditional probability must be applied

Multiplicative Principle for probabilities of independent events—if A and B are independent events, then

$$P(A \text{ and } B) = P(A \cap B)$$
$$= P(A) \times P(B)$$

4.5 Exercises

A   **1.** **Knowledge and Understanding** A truck driver, Andrea Ortiz, has a choice of routes as she travels among four cities. She can choose from four routes between Toronto and Oakville, two between Oakville and Hamilton, and three between Hamilton and Guelph. Find the total number of routes possible for complete Toronto–Oakville–Hamilton–Guelph trips.

2. A test has four true/false questions.
(a) Draw the tree diagram that shows all possible true/false combinations for the answers to the test.
(b) Determine the probability that a student will get all four correct by guessing.
(c) Determine the probability that a student will get three questions correct by guessing.

3. Suppose you conduct an experiment in which you draw a card from a standard 52-card deck, return it to the deck, and then draw another card. List the outcomes of the following event spaces.
(a) You get a seven of diamonds in both draws.
(b) You get an ace of spades on the first draw and a seven of diamonds on the second.
(c) You get a numbered club on both draws.
(d) You get an ace on the first draw and an even club on the second.

B **4. Communication** The faces of a standard six-sided die represent the numbers 1 through 6. Two of these dice, one red and the other white, are rolled simultaneously.
  (a) If you were to draw a tree diagram showing all possible combinations of the two dice, how many branches would it have?
  (b) Explain why there is only one combination for which the sum is 2.
  (c) Explain why there are two possible combinations for which the sum is 3.
  (d) Complete a chart like the one below for all possible sums of the two dice.

Sum	Required Rolls (Red, White)
2	(1, 1)
3	(1, 2); (2, 1)
4	

  (e) Find the probability of each possible sum.

5. A standard deck of cards has had all the face cards (jacks, queens, and kings) removed so that only the ace through ten of each suit remain. A game is played in which a card is drawn from this deck and a six-sided die is rolled. For the purpose of this game, an ace is considered to have a value of 1.
  (a) Determine the total number of possible outcomes for this game.
  (b) Describe the elements of the sample space for this experiment.
  (c) Find the probability of each of these events.
     (i) an even card and an even roll of the die
     (ii) an even card and an odd roll of the die
     (iii) a card of 3 and a roll of the die of 3 or less
     (iv) the sum of the card and the die is 7
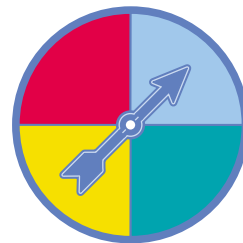   (v) the sum of the card and the die is 11

6. Suppose the two joker cards are left in a standard deck of cards. One of the jokers is red and the other is black. A single card is drawn from the deck of 54 cards, returned, and then a second card is drawn. Determine the probability of drawing

   (a) one of the jokers on the first draw and an ace on the second
   (b) the red joker on the second draw and a numbered card of any suit on the first
   (c) a queen on both draws
   (d) any black card on both draws
   (e) any numbered card less than 10 on the first draw and a card with the same number on the second
   (f) the red joker or a red ace on either draw

7. **Application** An airplane can make a safe landing if at least half of its engines are working properly. Suppose that engine failures are independent events. Determine whether a two-engine plane is safer than a four-engine plane if the chance that an engine fails is 1 in 2.

8. A health and safety committee is to be selected from all the people who work at a local factory. The committee is to consist of four members selected randomly from a list of ten names submitted by the shop leader. The list has the names of five union members and five workers who are not union members.

   (a) What is the probability that the first person selected from the list is a union member?
   (b) What is the probability that the first two people selected from the list are union members?
   (c) What is the probability that all the committee members are union members?
   (d) What is the probability that three of the four committee members are union members?

9. A paper bag contains a mixture of 3 types of candy. There are ten chocolate bars, seven fruit bars, and three packages of toffee. Suppose a game is played in which a candy is randomly taken from the bag, replaced, and then a second candy is drawn from the bag. If you are allowed to keep the second candy only if it was the same type as the one that was drawn the first time, calculate the probability of each of the following.

   (a) you will be able to keep a chocolate bar
   (b) you will be able to keep any candy
   (c) you won't be able to keep any candy

10. At the beginning of each month, the president of a small manufacturing company decides whether to spend $1000 or $2000 on advertising for the month. Suppose she makes her decision by tossing a coin.
    (a) What is the probability that she will spend $3000 on advertising during the first three consecutive months of this year?
    (b) What is the probability that she will spend more than $4000 on advertising during the first three consecutive months of this year?
    (c) What is the probability that she will spend more than $3000 on advertising during the first three consecutive months of this year?

11. **Knowledge and Understanding** A lottery uses a clear plastic drum that contains 100 ping-pong balls numbered from 0 to 9. There are 10 of each number in the drum. A ball is drawn from the drum and then returned after its number is written down. The process is repeated three times. The drawn numbers are used in their order of drawing to create a three-digit number.
    (a) Determine how many three-digit numbers can be created in this way.
    (b) Determine the probability that any number drawn in this way will end in a 5.
    (c) Determine the probability that any number drawn in this way will start with either a 1 or a 2.

12. Postal codes for Canada have the form LDL DLD, where L is any letter from A to Z, and D is any digit from 0 to 9. Some letters may not be permitted in certain positions of the postal code by Canada Post. As a result, the actual number of allowable postal codes will be different from the total number possible.
    (a) Estimate the total number of possible postal codes available for use in Canada.
    (b) Postal codes for Toronto start with the letter M. What is the probability that a postal code selected randomly will represent an area in Toronto?
    (c) Suppose you set up a computer program to generate postal codes randomly in a way such that no two generated postal codes are dependent on each other. What is the probability that your program will generate the same postal code two times in a row?

13. **Thinking, Inquiry, Problem Solving** A basketball player has a success rate of 80% for shooting free throws.
    (a) Calculate the following probabilities.
        (i) She will make three out of five attempts.
        (ii) She will miss five attempts in a row.
        (iii) She will make the first three attempts and miss the next two.
        (iv) She will make at least three out of five attempts.
    (b) Create a computer simulation for the situation in part (a) and use it to find the experimental probability for each event. Evaluate the quality of your simulation by comparing its results to the theoretical probabilities in part (a).

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

14. **Knowledge and Understanding** Draw a tree diagram to list the outcome set for an experiment in which a die is rolled, and then a four-coloured equal-probability spinner (red, blue, green, yellow) is spun. Find each probability.
    (a) a 6 is rolled and green is spun
    (b) an even number is rolled or red is spun

15. **Application** Joshua and Ravinder are having a video-game competition. Over the course of one week, they play several times. They have decided that the winner will be declared when someone wins three out of five games. Ravinder has had the game longer and his probability of winning each time is $\frac{3}{5}$. Joshua's probability of winning each time is $\frac{2}{5}$. Determine the probability that Joshua will win the weekly competition.

16. **Thinking, Inquiry, Problem Solving** Show that A and B are independent events if, and only if, $P(A \mid B) = P(A \mid B')$.

17. **Communication** Explain the difference between independent and dependent events. Use an example in your explanation.

Chapter Problem
Analyzing a Traditional Game

CP9. Make an outcome table or a tree diagram to model the game.
CP10. How many outcomes are possible if the six counters are distinguishable? Indistinguishable?
CP11. How many ways can the counters produce results that end up in
    (a) 5 points awarded?
    (b) 1 point awarded?
    (c) 0 points awarded?
CP12. Determine the probability that a player earns
    (a) 5 points on a turn
    (b) 1 point on a turn
    (c) 0 points on a turn
CP13. Based on these probabilities, does the point structure seem reasonable? Explain.