4.3 Finding Probability Using Sets

COUNTING OUTCOMES WITH VENN DIAGRAMS

Venn diagram—a diagram in which sets are represented by shaded or coloured geometrical shapes

A **Venn diagram**—named after John Venn (1834–1923), a British priest and logician—can be used to graphically describe the relationships between possible results of an experiment (or survey). **Compound events** are shown as combinations of simpler events. All the events exist within the larger collection of all possible outcomes of the experiment. This large set is called the sample space for the experiment. The letter S commonly represents the sample space.

The Master and Fellows of Gonville and Caius College, Cambridge

compound event—consists of two or more simple events

Venn Diagrams and Set Terminology

The following terminology and symbols are used in working with sets.

Intersection of Sets

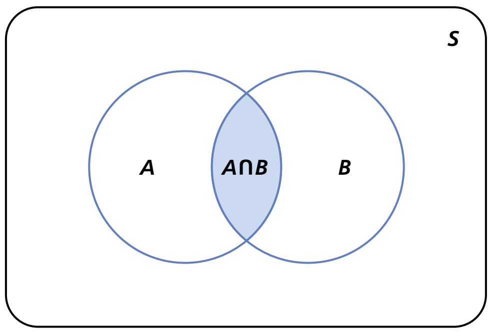
Given two sets, A and B, the set of common elements is called the *intersection* of A and B, and is written as $A \cap B$.

These common elements are members of set A and are also elements of set B. Consequently,

$$A \cap B = \{\text{elements in both } A \text{ AND } B\}$$

subset—a set whose members are all members of another set

The set $A \cap B$ is represented by the region of overlap of the two sets in the Venn diagram to the right. Sets A and B exist as sets within the larger set S. They are **subsets** of the set S.

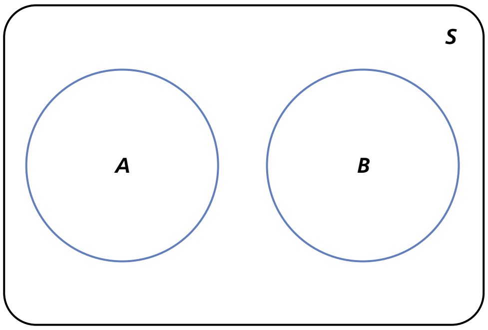


disjoint—two sets that have no elements in common

Disjoint Sets

If A and B have no elements in common (i.e., $n(A \cap B) = 0$), they are said to be **disjoint** and their intersection is the *empty set*, represented by the Greek letter $\varnothing$ (i.e., $A \cap B = \varnothing$).

The Venn diagram to the right shows disjoint sets A and B.



union—the set containing all of the elements in A as well as B

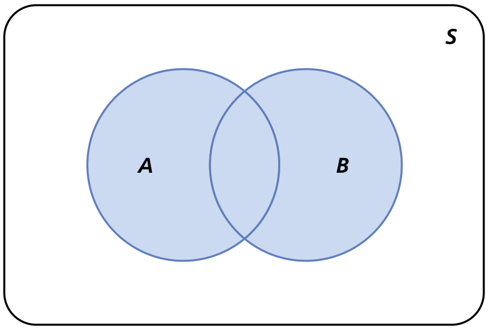
Union of Sets

The set formed by combining the elements of A with those in B is called the **union** of A and B, and is written as $A \cup B$.

The elements in $A \cup B$ are elements of A **or** they are elements of B. Consequently,

$$A \cup B = \{\text{elements in } A \text{ OR } B\}$$

The set $A \cup B$ is represented by the shaded area in the Venn diagram to the right.



Think about The Sample Space

What is the value of $P(\varnothing)$? Of $P(\varnothing')$?

Discussion Questions

1. Why is $n(\varnothing') = n(S)$?
2. Why is $n(S') = n(\varnothing)$?
3. If A and B are disjoint, why is $n(A \cup B) = n(A) + n(B)$?
4. If A and B are not disjoint, why is $n(A \cup B) < n(A) + n(B)$?

Example 1 Using a Venn Diagram to Solve a Counting Problem

Suppose a survey of 100 Grade 12 mathematics students in a local high school produced the following results.

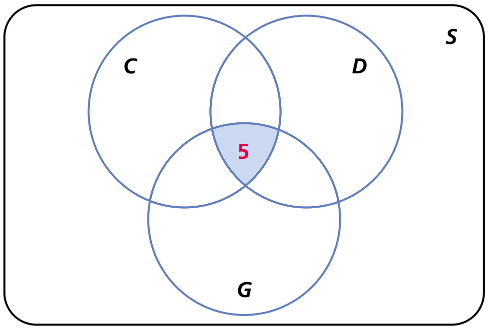
Math Course Taken	Number of Students
Advanced Functions and Introductory Calculus (AFIC)	80
Geometry and Discrete Math	33
Data Management	68
Geometry and Discrete Math and AFIC	30
Geometry and Discrete Math and Data Management	6
Data Management and AFIC	50
All three courses	5

How many students are enrolled in AFIC and in no other mathematics course? How many students are enrolled in AFIC? Or Data Management?

Solution

Construct a Venn diagram to represent the three groups of students. For convenience, we can label the sets as C for AFIC, D for Data Management, and G for Geometry and Discrete Math. The entire sample space, S, will consist of all the students in Grade 12.

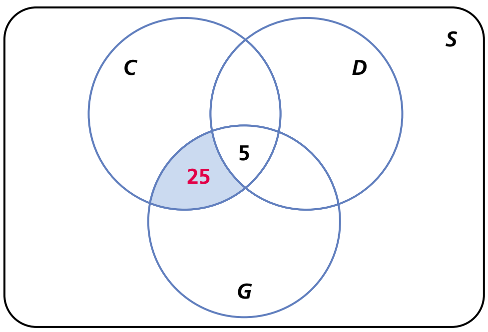
We will use the information in the chart to fill in each region of the Venn diagram. To avoid double-counting, you should start entering information from the chart in the very middle of the diagram and work your way out.



$n(C \cap G \cap D) = 5$, since there are five students in all three courses.

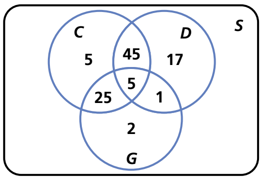
$n(G \cap C) = 30$

Since $n(C \cap G \cap D) = 5$, the number of students who take AFIC and Geometry and Discrete Math, but not Data Management, must be 25.



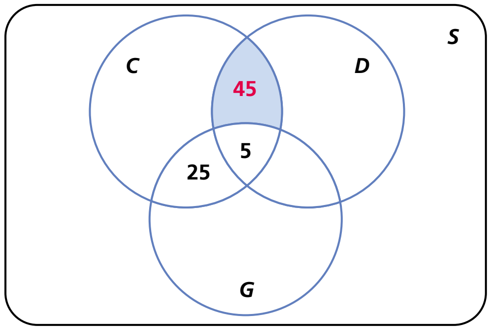
Think about Completing the Diagram

How could you use the information to complete all the remaining sections of the diagram? Remember, there are 100 students enrolled in Grade 12 math.

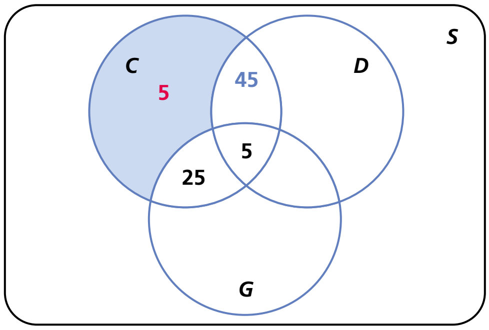


$n(D \cap C) = 50$

Since $n(C \cap G \cap D) = 5$, the number of students who take AFIC and Data Management, but not Geometry and Discrete Math, must be 45.

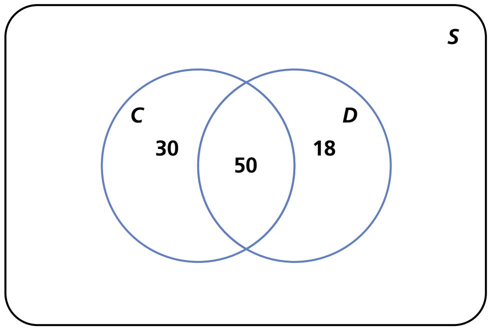


There were 80 students enrolled in AFIC, of which 75 have been accounted for. There must be, therefore, five students who take only AFIC and no other mathematics course.



The complete Venn diagram for the math course example is shown on the previous page. If we consider only the students taking AFIC and Data Management, the Venn diagram will have only two sets and one intersection.

The total number of students in both courses is 98.



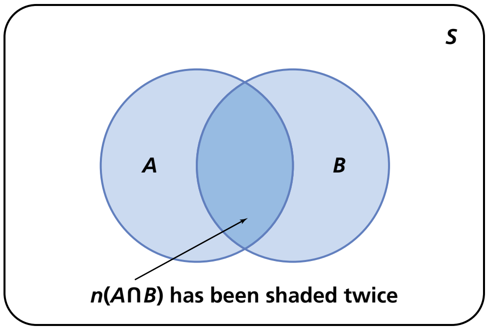
$$\begin{aligned} n(C \cup D) &= n(C) + n(D) - n(C \cap D) \\ &= 80 + 68 - 50 \\ &= 98 \end{aligned}$$

ADDITIVE PRINCIPLE FOR UNIONS OF TWO SETS

The solution to the counting problem above employed a Venn diagram. The counting strategy that was used leads to a more general counting strategy for unions of sets, called the **Additive Principle for unions of two sets**.

Additive Principle for Unions of Two Sets

Given two sets, A and B, the number of elements in $A \cup B$ can be found by totalling the number of elements in both sets and then subtracting the number that have been counted twice. The double-counted elements will be found in the intersection of the two sets.



$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Think about Overlap

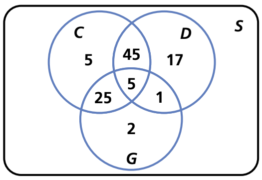
- Why is it necessary to subtract $n(A \cap B)$ in the general formula?
- Under what conditions would $n(A \cup B) = n(A) + n(B)$?

Example 2 Applying the Additive Principle

If you were asked to randomly select a student from the group of students described in Example 1, what is the probability that

(a) the student selected is enrolled only in AFIC?

(b) a student was in AFIC *or* in Data Management?



Solution

(a) The event of interest is the selection of a student who takes AFIC and no other course. Using the probability notation introduced earlier,

$$P(\text{student is in AFIC only}) = \frac{\text{number of students in AFIC only}}{\text{total number of students}}$$
$$= \frac{n(\text{AFIC only})}{n(S)}$$
$$= \frac{5}{100} = \frac{1}{20}$$

(b) This will include students enrolled in both courses or in only one of the courses. As a result,

$$P(\text{student is in AFIC } or \text{ in Data Management}) = P(C \cup D)$$
$$= \frac{n(C \cup D)}{n(S)}$$

Using the Venn diagram, we see the following:

- Since $n(C \cap D) = 50$, $P(C \cap D) = \frac{n(C \cap D)}{n(S)} = \frac{50}{100}$
- Since $n(C) = 80$, $P(C) = \frac{n(C)}{n(S)} = \frac{80}{100}$
- Since $n(D) = 68$, $P(D) = \frac{n(D)}{n(S)} = \frac{68}{100}$

Therefore, using the Additive Principle,

$$
\begin{aligned}
P(C \cup D) &= \frac{n(C \cup D)}{n(S)} \\
&= \frac{n(C) + n(D) - n(C \cap D)}{n(S)} \\
&= \frac{n(C)}{n(S)} + \frac{n(D)}{n(S)} - \frac{n(C \cap D)}{n(S)} \\
&= P(C) + P(D) - P(C \cap D) \\
&= \frac{80}{100} + \frac{68}{100} - \frac{50}{100} \\
&= \frac{98}{100} = \frac{49}{50}
\end{aligned}
$$

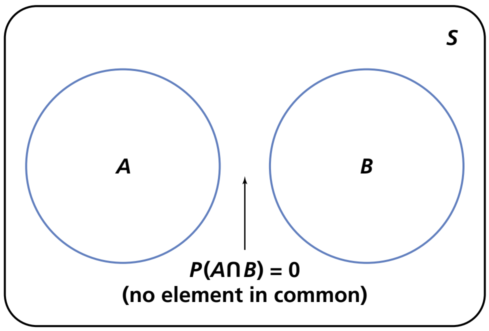
Additive Principle

This principle can also be applied to probabilities.

Additive Principle: Probability of the Union of Two Events

Given two events, A and B, the probability of the event in which A or B occurs is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



If A and B have no outcomes in common, they are said to be **mutually exclusive events** and $P(A \cup B) = P(A) + P(B)$.

mutually exclusive events—A and B are mutually exclusive events if $A \cap B = \varnothing$ and, as a result, $P(A \cup B) = P(A) + P(B)$ since $P(A \cap B) = P(\varnothing) = 0$.

Example 3 Additive Principle for Probabilities

If two dice are rolled, one red and one green, find the probability that a total of

(a) 2 or a total of 12 will occur **(b)** 4 or a pair will occur

D_2 \ D_1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Solution

(a) Let A be the event of rolling a total of 2 and B be the event of rolling a total of 12. Then,

$$P(A) = \frac{1}{36} \text{ and } P(B) = \frac{1}{36}$$

D_2 \ D_1	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

These events are mutually exclusive.

Thus, $P(A \cup B) = P(A) + P(B)$

$$= \frac{1}{36} + \frac{1}{36}$$

$$= \frac{1}{18}$$

(b) Let A be the event of rolling a total of 4 (red circle) and B (green circle) be the event of rolling a pair. Then,

$$P(A) = \frac{3}{36} \quad \text{and} \quad P(B) = \frac{6}{36}$$

$$= \frac{1}{12} \qquad\qquad = \frac{1}{6}$$

However, these events are not mutually exclusive. The outcome (red 2, green 2) is accounted for in both probabilities, so $P(A \cap B) = \frac{1}{36}$.

Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{36} + \frac{6}{36} - \frac{1}{36}$$

$$= \frac{2}{9}$$

KEY IDEAS

Venn diagram—a diagram in which sets are represented by geometrical shapes

compound event—consists of two or more simple events

intersection of sets—the set of common elements in two sets, A and B

$$A \cap B = \{\text{elements in both } A \text{ AND } B\}$$

disjoint sets—two sets with no elements or outcomes in common

union of sets—the set formed by combining the elements of A with those in B

$$A \cup B = \{\text{elements in both } A \text{ OR } B\}$$

Additive Principle for unions of two sets—

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Additive Principle for probabilities—

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$, if A and B are not mutually exclusive

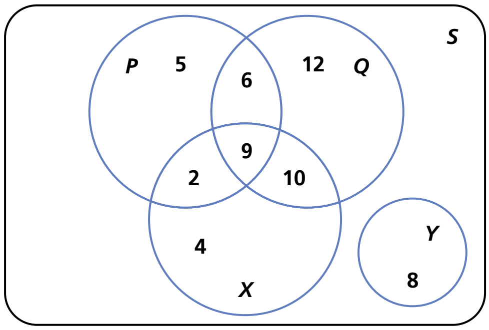
$P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive, because $P(A \cap B) = 0$

mutually exclusive events—A and B are mutually exclusive events if $A \cap B = \varnothing$ (no elements in common) and $n(A \cup B) = n(A) + n(B)$

4.3 Exercises

A

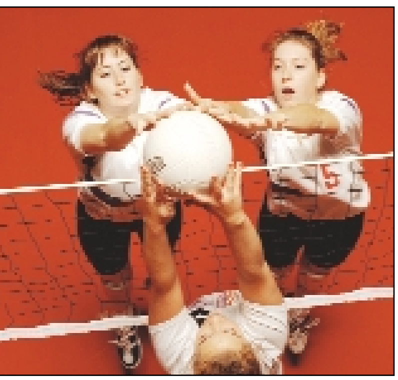
1. Using the Venn diagram below, list the elements (numbers) found in each of the following sets.



(a) $P \cap Q$ **(b)** $P \cup Q$ **(c)** $X \cap Q$
(d) $X \cup Q$ **(e)** $Y \cap Q$ **(f)** $P \cap Q \cap X$

2. For each of the following, find the indicated probability and state whether A and B are mutually exclusive.
(a) $P(A) = 0.5$, $P(B) = 0.2$, $P(A \cup B) = 0.7$, $P(A \cap B) = ?$
(b) $P(A) = 0.7$, $P(B) = 0.2$, $P(A \cup B) = ?$, $P(A \cap B) = 0.15$
(c) $P(A) = 0.3$, $P(B) = ?$, $P(A \cup B) = 0.9$, $P(A \cap B) = 0$

3. A sample space contains only three simple events: A, B, and C. If $P(A) = 0.2$ and $P(B) = 0.3$, find
(a) $P(A \text{ and } B)$ if A and B are mutually exclusive
(b) $P(A \text{ or } B)$ if A and B are mutually exclusive
(c) $P(C)$ if A and B are mutually exclusive



4. The probability that Kelly will make the volleyball team is $\frac{2}{3}$ and the probability that she will make the field hockey team is $\frac{3}{4}$. If the probability that she makes both teams is $\frac{1}{2}$, what is the probability that she makes at least one of the teams?

5. An aquarium at a pet store contains 20 guppies (12 females and 8 males) and 36 tetras (14 females and 22 males). If the clerk randomly nets a fish, what is the probability that it is a female or a tetra?

B

6. **Knowledge and Understanding** A paper bag contains a mixture of three types of candy. There are ten chocolate bars, seven fruit bars, and three packages of toffee.

(a) Draw a Venn diagram to illustrate the contents of the bag.

(b) Suppose a child selects one item from the bag at random. Determine the probability that the child chooses

(i) a chocolate bar

(ii) a package of toffee

(iii) something other than a fruit bar

7. An automobile manufacturer estimates the probability of a mechanical defect in the one-year warranty period is 0.65. The probability of any other defect is 0.35. The probability of encountering both types of defect is 0.20. What is the probability of encountering any type of defect?

8. At the start of flu season, Dr. Anna Ahmeed examines 50 patients over two days. Of those 50 patients, 30 have a headache, 24 have a cold, and 12 have neither symptom. Some patients have both symptoms.

(a) Draw a Venn diagram and determine the number of patients that have both symptoms.

(b) Find the probability that a patient selected at random

(i) has just a headache

(ii) has a headache or a cold

(iii) does not have cold symptoms

9. Find the probability that, when you draw a single card from a well-shuffled standard deck of 52 playing cards, you choose a 9 or a 10.

10. Find the probability that, when you draw a single card from a well-shuffled standard deck of 52 playing cards, you choose an ace or a club.

11. Find the probability that when you roll two dice, the sum of the outcomes is greater than 6 or you get a 5 on one of the dice.

12. The probability it will rain today is 0.4 and the probability it will rain tomorrow is 0.3. The probability it will rain both days is 0.2. What is the probability it will rain today or tomorrow?

13. **Communication** If events A and B are mutually exclusive, explain why $P(A \cup B)$ is the sum of the probabilities of each event. Use an example in your explanation.

14. **Application** In a group of 45 students, 28 have dark hair, 19 are taller than 185 cm, and 5 neither have dark hair nor are taller than 185 cm. Some have dark hair and are taller than 185 cm. If a student is selected at random, determine the probability that the student is

(a) taller than 185 cm and has dark hair

(b) taller than 185 cm or has dark hair

(c) not taller than 185 cm

15. **Thinking, Inquiry, Problem Solving** In 2001, your company paid overtime wages or hired temporary help during 32 weeks of the year. Overtime was paid during 26 weeks and temporary help was hired during 15 weeks. If at year's end an auditor checks your accounting records and randomly selects one week to check the company's payroll, what is the probability that the auditor will select a week in which you paid overtime wages and had hired temporary help?

16. Give an example of two events that are not mutually exclusive and create and solve a probability question using the two events you chose.

ADDITIONAL ACHIEVEMENT CHART QUESTIONS

17. **Knowledge and Understanding** Jarvis High School has 1500 students. The first school dance of the year was attended by 740 students, while only 440 attended the second dance. If 285 students attended both, how many did not go to either dance?

18. **Communication** Give an example and describe a situation in which two events are
    (a) mutually exclusive
    (b) not mutually exclusive

19. **Application** An advertiser is told that 70% of all adults in the Greater Toronto Area (GTA) read *The Toronto Star* and 60% watch CityTV. She is also told that 40% do both: read *The Toronto Star* and watch CityTV. If she places an advertisement in *The Toronto Star* and runs a commercial on CityTV, what is the probability that a person selected at random in the GTA will see at least one of these?

20. **Thinking, Inquiry, Problem Solving** If A, B, and C are three events that are not mutually exclusive, state the Additive Principle for the union of these three sets.