# 4.2 Theoretical Probability

The study of probability began with the analysis of games of chance by the mathematicians Cardano, Galileo (pictured), Pascal, and Fermat. When you state the probability of an event, you are making a statement about the likelihood of that event occurring.

How do you find the probability of an event without performing an experiment?

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## GENERAL DEFINITION OF PROBABILITY

The previous section introduced you to the concept of experimental probability. Actual data were used to determine the relative frequency of a particular event.

Probability is often used to predict the likelihood that a particular event will occur. Experimental probabilities or relative frequencies determined from surveys only give an estimate of the likelihood that a particular event will occur.

It is possible to determine a more accurate probability for some events, such as rolling a 4 on a die. For example, in rolling a die six times, 4 may show up twice, so the experimental probability is $\frac{2}{6}$.

However, we know that there are six possible outcomes when a die is rolled. Only one of these outcomes is the event of rolling a 4. This is an example of a **simple event**. Since the possible outcomes are all equally likely to happen, it is reasonable to expect that the fraction of the time you roll a 4 is the ratio of the number of ways a 4 can occur to the number of possible outcomes.

**simple event**—an event that consists of exactly one outcome

$$P(\text{rolling a } 4) = \frac{1}{6}$$

The resulting value is called the **theoretical probability**. Given a large enough number of trials, the theoretical probability and the experimental probability should be approximately equal.

**theoretical probability**—when all the outcomes of an experiment that correspond to an event are equally likely, the probability of the event is the ratio of the number of outcomes that make up that event to the total number of possible outcomes

**Think about Probability**

- Why is the minimum value of a probability 0?
- Why is the maximum value of a probability 1?
- Probability is often expressed as a ratio. What other ways can probability be expressed?

**sample space**—the collection of all possible outcomes of the experiment

**event space**—the collection of all outcomes of an experiment that correspond to a particular event

## Theoretical Probability

Assuming that all outcomes are equally likely, the probability of an event in an experiment is the ratio of the number of outcomes that make up that event to the total number of possible outcomes.

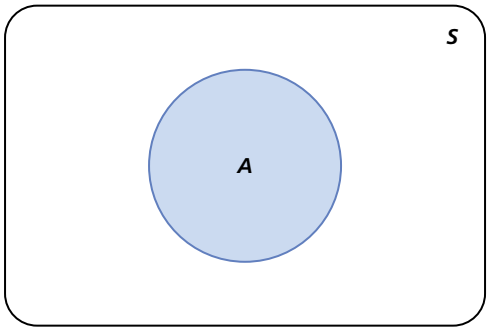
The formula for the probability of an event $A$ is

$$P(A) = \frac{n(A)}{n(S)}$$

where

- $S$ is the collection of all possible outcomes of the experiment—the **sample space**;
- $A$ is the collection of outcomes that correspond to the event of interest—the **event space**; and
- the mathematical notation $n(A)$ means the "number of elements in the set $A$." Both $n(A)$ and $n(S)$ are the numbers of elements in the two sets.

A Venn diagram can be used to show the relationship between the event space, $A$, and the sample space, $S$. The use of Venn diagrams as possible solution tools will be explored in Section 4.3 on page 221.




### Example 1 Choosing Coloured Marbles From a Bag

A bag contains five red marbles, three blue marbles, and two white marbles. What is the probability of drawing a blue marble?

### Solution

The event of selecting a blue marble consists of three possible outcomes (one for each blue marble). Therefore,

$$P(\text{blue}) = \frac{\text{number of blue marbles}}{\text{number of marbles in the bag}}$$
$$= \frac{n(A)}{n(S)}$$
$$= \frac{3}{10}$$

### Example 2 Rolling a Die

If a single die is rolled, determine the probability of rolling

**(a)** an even number **(b)** a number greater than 2

#### Solution

**(a)** A die has six sides. Each side has a series of dots that represent the numbers 1 to 6. The event of rolling an even number contains the outcomes 2, 4, and 6.

$$P(\text{even number}) = \frac{\text{number of even numbers}}{\text{total number of possible outcomes}}$$
$$= \frac{3}{6} = \frac{1}{2}$$

**(b)** $$P(\text{number} > 2) = \frac{\text{number of numbers greater than 2}}{\text{total number of possible outcomes}}$$
$$= \frac{4}{6} = \frac{2}{3}$$

## Probability and Complementary Events

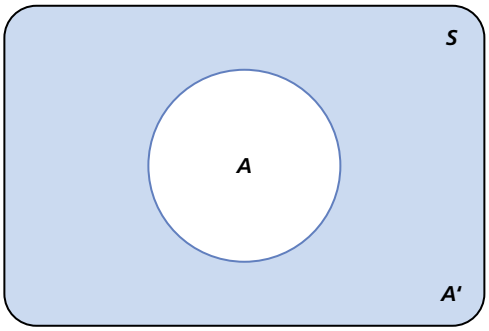
In Example 2, $P(\text{number} > 2) = \frac{2}{3}$. What is $P(\text{number} \le 2)$? The event of a number $\le 2$ contains all outcomes in the sample space that are not in $A$ (number $> 2$). These numbers are in the complement of $A$ and are represented by $A'$.

### Probability and Complementary Events

The *complement* of a set, $A$, is written as $A'$ and consists of all the outcomes in the sample space that are not in set $A$.

$$A' = \{\text{outcomes in } S \text{ that are NOT in } A\}$$

The Venn diagram shows $A'$ as the shaded region within $S$ that is entirely outside of $A$.




### Probability of a Complementary Event

If $A$ is an event in a sample space, the probability of the complementary event, $A'$, is given by

$$P(A') = 1 - P(A)$$

**Think about**
**The Sample Space**

- What is the complement of the entire sample space $S$?
- What is the value of $P(S)$? Of $P(S')$?

### Example 3 Drawing an Ace From a Standard Deck of Cards

A standard deck of cards comprises 52 cards in four suits—clubs, hearts, diamonds, and spades. Each suit consists of 13 cards—ace through 10, jack, queen, and king.

What is the probability of drawing an ace from a well-shuffled deck? What is the probability of drawing anything but an ace?

**Solution**

The drawing of any single card from the deck is as likely as the drawing of any other card. There are 4 aces in a deck of 52 cards. The sample space has 52 elements and there are 4 outcomes that correspond to drawing an ace.

Therefore,

$$P(\text{ace}) = \frac{\text{number of aces in the deck}}{\text{number of cards in the deck}}$$
$$= \frac{4}{52}$$
$$= \frac{1}{13}$$
$$P(\text{not an ace}) = 1 - P(\text{ace})$$
$$= 1 - \frac{1}{13}$$
$$= \frac{12}{13}$$

**KEY IDEAS**

**sample space**—the collection of all possible outcomes of an experiment

**event space**—the collection of all outcomes of an experiment that correspond to a particular event

**simple event**—an event that consists of exactly one outcome

**theoretical probability**—the ratio of the number of outcomes that make up that event to the total number of possible outcomes

$$P(A) = \frac{n(A)}{n(S)}$$

where

- $S$ is the collection of all possible outcomes of the experiment;
- $A$ is the collection of outcomes that correspond to the event of interest; and
- $n(A)$ and $n(S)$ are the numbers of elements in the two sets.

**probability of a complementary event**—if $A$ is an event in a sample space, the probability of the complementary event, $A'$, is given by $P(A') = 1 - P(A)$

## 4.2 Exercises

**A**

**1.** Suppose you conduct an experiment in which you draw a card from a standard 52-card deck. List the outcomes in the following event spaces.
- **(a)** you draw a seven of diamonds
- **(b)** you draw an ace
- **(c)** you draw a numbered club
- **(d)** you draw an even-numbered card of any suit

**2.** **(a)** Which of the events in Question 1 can be classified as simple events?
**(b)** Compute the theoretical probability of each event in Question 1.

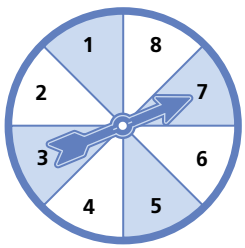
**3.** **Knowledge and Understanding** Three black marbles and two red marbles are in a box. One marble is secretly drawn from the box.
- **(a)** What is the total number of possible outcomes?
- **(b)** What is the probability that the marble selected is black?
- **(c)** What is the probability that the marble selected is red?
- **(d)** What is the probability that the marble selected is neither red nor black?

**4.** Suppose the two joker cards are left in a standard deck of cards. One of the jokers is red and the other is black. A single card is drawn from the deck of 54 cards. Determine the probability of drawing
- **(a)** one of the jokers
- **(b)** the red joker
- **(c)** a queen
- **(d)** any black card
- **(e)** any card less than 10 (an ace has a value of one)
- **(f)** the red joker or a red ace

**5.** Determine the theoretical probability for each of the following events.
- **(a)** getting tails with a coin toss
- **(b)** rolling a 3 on a die
- **(c)** drawing a red card from a well-shuffled deck
- **(d)** drawing a black seven from a well-shuffled deck
- **(e)** drawing anything but a face card from a well-shuffled deck

**B**

**6.** A spinner is divided into eight equal sectors, numbered 1 through 8.




- **(a)** What is the probability of spinning an odd number?
- **(b)** What is the probability of spinning a number divisible by 4?
- **(c)** What is the probability of spinning a number less than 3?

**7.** A bag contains 12 identically shaped blocks, 3 of which are red and the remainder are green. The bag is well-shaken and a single block is drawn.
**(a)** What is the probability that the block is red?
**(b)** What is the probability that the block is not red?

**8.** Each of the letters for the word MATHEMATICS is printed on same-sized pieces of paper and placed in a hat. The hat is shaken and one piece of paper is drawn.
**(a)** What is the probability that the letter S is selected?
**(b)** What is the probability that the letter M is selected?
**(c)** What is the probability that a vowel is selected?

**9.** **Communication** In the game of Bingo, the numbers 1 to 75 are marked on balls and drawn from a container that constantly mixes the balls. Fifteen balls, numbered 1 to 15, are marked with a B; the next 15 with an I; and so on.

**(a)** What is the probability that an O is drawn first?
**(b)** What is the probability that a multiple of 5 is drawn first?
**(c)** Which is more likely to occur when the first ball is drawn, an even number or an odd number? Justify your answer.

**10.** Three coins are tossed at the same time. Find the probability that
**(a)** all come up heads
**(b)** at least one comes up tails
**(c)** exactly two come up heads

**11.** **Application** A colour TV is given as a door prize at a dance. A total of 360 tickets are sold and are numbered 1–360. If the winning ticket is drawn at random, find the probability that the winning ticket is between 220 and 280 (excluding 220 and 280).

**12.** Two dice are rolled: one is red and the other is white.
**(a)** How many outcomes are possible?
**(b)** Determine the probability that the sum of the two dice totals 7.
**(c)** Determine the probability that the sum of the two dice totals anything but 7.

**13.** You are dialing a friend's phone number but can't remember the last digit. If you guess at the final digit, what is the probability that you dialed the number correctly?

**C** **14.** **Thinking, Inquiry, Problem Solving** A picnic cooler contains different types of cola: 12 regular, 8 cherry, 10 diet, 6 diet cherry, 8 caffeine-free, and some caffeine-free diet. You pick a can of cola without looking at its type. There is a 44% chance that the drink selected is diet. How many caffeine-free diet colas are in the cooler?

**15.** Studies show that when people are asked to choose one of the integers 1, 2, 3, or 4, more than half of the people choose the same number. Conduct this experiment in your class and determine which number was chosen more often. Explain why asking a person to choose a number is not the same as randomly selecting a number.

## ADDITIONAL ACHIEVEMENT CHART QUESTIONS

**16.** **Knowledge and Understanding** Sarah has five blues CDs, four rap CDs, and nine alternative CDs in the console of her car. If she reaches in and grabs a CD, what is the probability that it is a rap CD?

**17.** **Communication** What is the difference between an outcome and an event? Use an example in your explanation.

**18.** **Application** Four students—Mark, Barry, Rita, and Francine—must present their class projects today. The teacher puts their names in a hat and selects the order in which they will present. Determine the probability that the boys will start and end today's presentations.

**19.** **Thinking, Inquiry, Problem Solving** Find the probability that a number chosen at random between 1 and 100 is divisible by 2 or 7.

## Chapter Problem

### Analyzing a Traditional Game

**CP7.** What is the theoretical probability that one of the counters in the game turns up black? Red? What assumptions must you make when calculating these probabilities?

**CP8.** Based on what you know, is it possible to easily determine the theoretical probability of each outcome in the game? Why or why not?